Math. Struct. in Comp. Science (2011), *vol.* 21, *pp.* 267–320.
doi:10.1017/S0960129510000551

Traces for coalgebraic components

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Received 30 March 2010; revised 15 September 2010

This paper contributes a feedback operator, in the form of a monoidal trace, to the theory of coalgebraic, state-based modelling of components. The feedback operator on components is shown to satisfy the trace axioms of Joyal, Street and Verity. We employ McCurdy's tube diagrams, which are an extension of standard string diagrams for monoidal categories, to represent and manipulate component diagrams. The microcosm principle then yields a canonical 'inner' traced monoidal structure on the category of resumptions (elements of final coalgebras/components). This generalises an observation by Abramsky, Haghverdi and Scott.

1. Introduction

The subject of study in the field of coalgebra is state-based computation. A computer is a device that has an internal state, roughly given by the content of all its memory cells and registers, that is not directly observable. However, a user can observe and modify part of this state through I/O devices, such as a screen or keyboard. Very abstractly, such a computer is captured as a coalgebra $X \to F(X)$, where X represents the state and F captures the type of operations (for observation and modification) that one can perform on these states. A simple example is a deterministic automaton of the form $X \to (X \times B)^A$, where A is a type for input and B is a type for output.

The coalgebraic view on state-based systems yields a generic view, for instance, on bisimilarity (indistinguishability of states) and compositionality (see, for example, Turi and Plotkin (1997)), and on modal logic (see, for example, Kurz and Pattinson (2005)), giving a way to reason about properties of states with dynamic operators like 'nexttime'. Here we use coalgebras to (further) develop a calculus of *components*, which describes various ways of combining components (smaller subsystems) into larger systems. Numerous component calculi have been proposed, such as Reo (Arbab 2004), with the aim of aiding the modular design of complex systems. The existing component calculi come with different sets of (typically several) *component connectors*. In earlier work (Hasuo *et al.* 2009; Asada and Hasuo 2010), we have focused on a small core subset of such calculi, with sequential composition and parallel composition only. In this paper we add a feedback operator to this calculus, in the form of a trace.

This calculus may be understood as a many-sorted process algebra that acts directly on systems; by employing the *microcosm principle*, we also obtain process algebraic operators on behaviours (Hasuo *et al.* 2008; Hasuo *et al.* 2009).

A categorical approach to system composition based on bicategories was developed in Katis *et al.* (1997). The bicategorical aspect comes up if one uses components as morphisms, because composition of components is associative only up to isomorphism – see Lemma 4.2 (3). This approach was extended in Barbosa (2001; 2003) with monads to allow for different kinds of computation (Moggi 1991). Here, as in Hasuo *et al.* (2008) and Hasuo *et al.* (2009), we take a slightly different approach and use components as objects in a category, with fixed input and output. In order to deal with the relabelling of input and output, we need to organise the whole as an indexed category. However, this indexing is straightforward and poses no technical obstacles.

A crucial ingredient that has been missing so far in these calculi of components is feedback, which allows us to include 'loops' in diagrams of components so that we can capture recursive flows. In this paper we extend coalgebraic component calculi with such a feedback mechanism in the form of a trace operator. Traces as feedback operators were introduced abstractly in Joyal *et al.* (1996), and we, essentially, follow their framework, except that the required identities for these traces only hold up to isomorphism (just like for composition). The traces that we introduce for coalgebraic components are based on the trace construction in Kleisli categories from Jacobs (2010). In his thesis, Barbosa discusses a 'partial feedback' operator (see Barbosa (2001, Chapter 5, Section 51)) but it is not a proper trace operator (in the sense of Joyal *et al.* (1996)) because it does not have the correct type or behaviour.

It is very useful to have a diagrammatic language for components that allows us to build a composite system using a picture with lines representing connections between them. There is already a standard language of string diagrams for monoidal categories (see Penrose (1971) and Joyal and Street (1991)), and here we employ McCurdy's extension (McCurdy 2010), which we call *tube diagrams*, for capturing coalgebraic components and their connections. This language can be used to reason about coalgebraic components using specific diagrammatic manipulations – see Section 5. The additional (third) dimension obtained by using tubes rather than strings is needed because in our calculus input can be structured both multiplicatively (with tensor $\otimes$) and additively (with coproduct $+$). The feedback/trace operator works with respect to this additive structure.

The additional trace operator for components embodies *iteration* in data processing, which is a fundamental concept in computer science. Our theory is generic because it is parametrised by a monad T: the monad represents the computational effect that makes an iterated function 'total'. The prototypical example of such an effect is partiality (that is, $T = 1 + (-)$, the lift monad) but non-determinism and probability also fits in this setting. To demonstrate the versatility of our results, we derive the traced monoidal structure of the category of T-*resumptions*, in a canonical manner. This generalises the observation in Abramsky *et al.* (2002), where the authors focus on the partiality effect (that is, $T = 1 + (-)$) and the trace operator on resumptions is introduced in concrete terms. Here, instead, we derive trace operators on various resumptions uniformly from the trace operators on components in the style of Krstić *et al.* (2001). The derived traced monoidal category *lives in* the 'traced monoidal bicategory' of components, providing an instance of the *microcosm principle* (Baez and Dolan 1998; Hasuo *et al.* 2008; Hasuo *et al.* 2009). Another way to look at our construction is that it uniformly transforms the traced monoidal category of T-computations (namely the Kleisli category $\mathcal{K}\ell(T)$) into the traced monoidal category of T-*strategies* (identified with T-resumptions). By further applying the Int-construction (Joyal *et al.* 1996), one obtains the compact closed category of *(stateful) T-games*, as pointed out in Abramsky and Jagadeesan (1994).

The paper is organised as follows. After some preliminary material in Section 2, we introduce in Section 3 the fundamental operator of 'state extension', which adds a state object, either on the left or on the right, to a coalgebraic component. This will play an important role in the rest of the paper, for instance, in Section 4 on the various composition operators: sequential $\ggg$; multiplicative parallel $\|$; and additive parallel $\square$. In Section 5, we describe the tube calculus, with several distributivity results, and in Section 6, we define traces for components and prove that the trace axioms hold – this is the main contribution of the paper. Finally, in Section 7 we identify the category of resumptions as an 'inner' traced monoidal category in the category of components.

2. Preliminaries

The basic setting is described by a category $\mathbb{C}$ with some structure and a monad T on $\mathbb{C}$. We assume in the first place that $\mathbb{C}$ is a symmetric monoidal category with tensor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and tensor unit $I \in \mathbb{C}$, together with canonical isomorphisms:

$$X \otimes (Y \otimes Z) \xrightarrow[\cong]{\alpha} (X \otimes Y) \otimes Z \quad I \otimes X \xrightarrow[\cong]{\lambda} X \quad X \otimes Y \xrightarrow[\cong]{\gamma} Y \otimes X \qquad (1)$$

We will often write $\rho = \lambda \circ \gamma : X \otimes I \xrightarrow{\cong} X$.

We assume that the monad $T = (T, \eta, \mu)$ on $\mathbb{C}$ is symmetric monoidal (also known as commutative) through a natural transformation with components

$$\mathsf{dst} \colon T(X) \otimes T(Y) \to T(X \otimes Y)$$

interacting appropriately with the monoidal isomorphisms (1) and with the unit η and multiplication μ of the monad. For this natural transformation dst, we write

$$\mathsf{st} = \mathsf{dst} \circ (\eta \otimes \mathrm{id}) : X \otimes T(Y) \to T(X \otimes Y)$$

for the 'strength' map of the monad. The abbreviation 'dst' stands for 'double strength'.

The Kleisli category of the monad T is written as $\mathcal{K}\ell(T)$. We will usually write a fat dot $\bullet$ for composition in $\mathcal{K}\ell(T)$ if we wish to distinguish it from ordinary composition $\circ$ in $\mathbb{C}$. The inclusion functor $J : \mathbb{C} \to \mathcal{K}\ell(T)$, given by $X \mapsto X$ and $f \mapsto \eta \circ f$, is sometimes not written explicitly when the context allows us to omit it.

We also assume that the category $\mathbb{C}$ has (distributive) coproducts $+, 0$, also written as $\coprod$ when indexed over a set. The associated coprojections will be written as $\kappa_i : X_i \to X_1 + X_2$, and cotupling of $f_i : X_i \to Y$ is written as $[f_1, f_2] : X_1 + X_2 \to Y$. Distributivity here means that the tensor $\otimes$ distributes over coproducts $(0, +)$. In the binary case, this means that the canonical maps

$$\mathsf{dis} \stackrel{\text{def}}{=} \Big(X \otimes A + X \otimes B \xrightarrow{[\mathrm{id}_X \otimes \kappa_1, \mathrm{id}_X \otimes \kappa_2]} X \otimes (A+B) \Big) \tag{2}$$

are isomorphisms. Additionally, the canonical map $0 \to X \otimes 0$ is an isomorphism; this is the nullary case of distributivity. Notice that if the category $\mathbb{C}$ is monoidal closed, these isomorphisms exist automatically, since $X \otimes -$ is then left adjoint to exponentiation $X \multimap (-)$.

The following lemma lists some elementary equations for the distribution map dis that will be used later – the proofs are easy and left as an exercise.

Lemma 2.1. The distribution map dis defined in (2) satisfies

(1) naturality: $\mathsf{dis} \circ (f \otimes g + f \otimes h) = (f \otimes (g+h)) \circ \mathsf{dis}$.
(2) $\alpha \circ (\mathrm{id} \otimes \mathsf{dis}) \circ \mathsf{dis} = \mathsf{dis} \circ (\alpha + \alpha)$.
(3) $\lambda \circ \mathsf{dis} = \lambda + \lambda$.

There are also rules regulating the interaction of the distribution map (2) with the monoidal isomorphisms associated with coproducts. We shall label them with a $+$, as in α_+, in order to distinguish them from the isomorphisms for $\otimes$.

Lemma 2.2. The distributivity map dis interacts with the monoidal isomorphisms for $+$ as follows:

(1) Interaction with ρ_+:

$$\begin{tikzcd}
X \otimes A + 0 \arrow[rr, "\rho_+"] \arrow[d, "\mathrm{id}+!"'] & & X \otimes A \arrow[dd, equal] \\
X \otimes A + X \otimes 0 \arrow[d, "\mathsf{dis}"'] & & \\
X \otimes (A+0) \arrow[rr, "\mathrm{id}\otimes\rho_+"] & & X \otimes A
\end{tikzcd}$$

(2) Interaction with γ_+:

$$\begin{tikzcd}
X \otimes A + X \otimes B \arrow[r, "\gamma_+"] \arrow[d, "\mathsf{dis}"'] & X \otimes B + X \otimes A \arrow[d, "\mathsf{dis}"] \\
X \otimes (A+B) \arrow[r, "\mathrm{id}\otimes\gamma_+"] & X \otimes (B+A)
\end{tikzcd}$$

(3) Interaction with α_+:

$$\begin{tikzcd}
(X \otimes A) + ((X \otimes C) + (X \otimes D)) \arrow[r, "\alpha_+"] \arrow[d, "\mathrm{id}+\mathsf{dis}"'] & ((X \otimes A) + (X \otimes C)) + (X \otimes D) \arrow[d, "\mathsf{dis}+\mathrm{id}"] \\
(X \otimes A) + (X \otimes (C+D)) \arrow[d, "\mathsf{dis}"'] & (X \otimes (A+C)) + (X \otimes D) \arrow[d, "\mathsf{dis}"] \\
X \otimes (A + (C+D)) \arrow[r, "\mathrm{id}\otimes\alpha_+"] & X \otimes ((A+C)+D)
\end{tikzcd}$$

3. Coalgebraic components and state extension

This section describes the basic construction of what we call 'state extension' for coalgebraic components. It serves as an auxiliary operator for constructions like composition and trace in later sections. Formally, state extension is described as an action of a (monoidal) category on a category of components (Janelidze and Kelly 2001). This insight is not important for what follows, so it is elaborated separately in Section 3.1.

As in Barbosa (2001; 2003) and Hasuo *et al.* (2009), we consider coalgebraic components of the form

$$X \otimes A \xrightarrow{c} T(X \otimes B) \tag{3}$$

where X is the state space, A is the type/object of inputs and B is the type of outputs. In general we shall use letters like X, Y, Z, U, V for states and A, B, C for in/outputs. The monad T captures the type of computation involved, following the standard approach in monadic computation. For instance, non-deterministic when T=powerset, partial when T=lift, probabilistic when T=distribution, with side-effects when $T = (S \times -)^S$ with S for states, or even deterministic when T=identity. Throughout this article, T will be used as a parameter, and we will make it explicit when any additional requirements are needed. However, T will often be invisible when we are working in the Kleisli category $\mathcal{K}\ell(T)$ of T.

Strictly speaking, the map c in (3) is not a coalgebra. When the category $\mathbb{C}$ is closed, with $\multimap$ as exponent (internal hom), this map c may equivalently be written in coalgebra form $X \longrightarrow A \multimap T(X \otimes B)$. Using the form (3), we can work without the closedness assumption on the category $\mathbb{C}$.

For fixed objects $A, B \in \mathbb{C}$, we thus have a category of such coalgebraic components, which we shall write as $\mathbf{Comp}(T, A, B)$. A morphism

$$\Big(X \otimes A \xrightarrow{c} T(X \otimes B)\Big) \xrightarrow{\quad f \quad} \Big(Y \otimes A \xrightarrow{d} T(Y \otimes B)\Big) \tag{4}$$

in $\mathbf{Comp}(T, A, B)$ is a map $f : X \to Y$ in $\mathbb{C}$ satisfying

$$T(f \otimes \mathrm{id}_B) \circ c = d \circ (f \otimes \mathrm{id}_A).$$

There is thus an obvious forgetful functor $\mathbf{Comp}(T, A, B) \to \mathbb{C}$ that maps a coalgebraic component to its underlying state.

Each 'pure' map $f : A \to B$ in $\mathbb{C}$ gives rise to a coalgebraic component, written as $\mathtt{arr} f$ in $\mathbf{Comp}(T, A, B)$, with the tensor unit I as the trivial state space, as in

$$\mathtt{arr} f \stackrel{\text{def}}{=} \Big(I \otimes A \xrightarrow{\mathrm{id}_I \otimes f} I \otimes B \xrightarrow{\quad \eta \quad} T(I \otimes B)\Big). \tag{5}$$

For maps $g : C \to A$ and $h : B \to D$ in $\mathbb{C}$ there is an obvious relabelling functor

$$(g, h)^* : \mathbf{Comp}(T, A, B) \to \mathbf{Comp}(T, C, D)$$

given by

$$\Big(X \otimes A \xrightarrow{\quad c \quad} T(X \otimes B)\Big) \longmapsto \Big(X \otimes C \xrightarrow{\mathrm{id} \otimes g} X \otimes A \xrightarrow{\quad c \quad} T(X \otimes B) \xrightarrow{T(\mathrm{id} \otimes h)} T(X \otimes D)\Big).$$

On morphisms of coalgebraic components, relabelling is the identity. Components thus form a functor

$$\mathbf{Comp}(T, -, -) : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Cat},$$

which can be described as **Cat**-valued distributors/profunctors/arrows, as shown in Hasuo *et al.* (2008), Hasuo *et al.* (2009) and Asada and Hasuo (2010).

With these definitions in place, we can now introduce state extension.

Definition 3.1. For a coalgebraic component $X \otimes A \xrightarrow{c} T(X \otimes B)$ with state space X, as in (3), and for objects $U, V \in \mathbb{C}$, we define two new coalgebraic components $U \mid c$ and $c \mid V$, with extended state spaces $U \otimes X$ and $X \otimes V$, respectively, as follows:

$$\begin{tikzcd}
(U \otimes X) \otimes A \arrow[rr, "U|c"] \arrow[d, "\alpha^{-1}"', "\cong"] & & T((U \otimes X) \otimes B) \\
U \otimes (X \otimes A) \arrow[r, "\mathrm{id} \otimes c"] & U \otimes T(X \otimes B) \arrow[r, "\mathsf{st}"] & T(U \otimes (X \otimes B)) \arrow[u, "\cong", "T(\alpha)"']
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes V) \otimes A \arrow[rr, "c|V"] \arrow[d, "\alpha^{-1} \circ (\gamma \otimes \mathrm{id})"', "\cong"] & & T((X \otimes V) \otimes B) \\
V \otimes (X \otimes A) \arrow[r, "\mathrm{id} \otimes c"] & V \otimes T(X \otimes B) \arrow[r, "\mathsf{st}"] & T(V \otimes (X \otimes B)) \arrow[u, "\cong", "T((\gamma \otimes \mathrm{id}) \circ \alpha)"']
\end{tikzcd}$$

Clearly, $c \mid V = T(\gamma \otimes \mathrm{id}) \circ (V \mid c) \circ (\gamma \otimes \mathrm{id})$.

State extension can be defined more conveniently directly in the Kleisli category $\mathcal{K}\ell(T)$, namely as

$$\begin{aligned}
U \mid c &= \alpha \bullet (U \otimes c) \bullet \alpha^{-1} \\
d \mid V &= (\gamma \otimes \mathrm{id}) \bullet (V \mid d) \bullet (\gamma \otimes \mathrm{id}).
\end{aligned}$$

Also, reasoning about these constructions is easier when done directly in the Kleisli category. In the proof of the main lemma below, we shall use a mix of reasoning in both $\mathbb{C}$ and $\mathcal{K}\ell(T)$ to demonstrate the difference. Later on in this paper, we will reason mostly in the Kleisli category.

State extension satisfies the following basic properties.

Lemma 3.2. The above operators $U \mid c$ and $c \mid V$ satisfy the following properties.

(1) $U \mid \eta = \eta$ and $\eta \mid V = \eta$.
(2) $U \mid (\mathsf{arr} f) = \eta \circ (\mathrm{id}_{(U \otimes I)} \otimes f)$ and $(\mathsf{arr} f) \mid V = \eta \circ (\mathrm{id}_{(I \otimes U)} \otimes f)$, with arr defined in the definition (5) above.
(3) For two composable components

$$\begin{aligned}
X \otimes A &\xrightarrow{c} T(X \otimes B) \\
X \otimes B &\xrightarrow{d} T(X \otimes C)
\end{aligned}$$

with the same state space, state extension commutes with Kleisli composition $\bullet$, in the sense that:

$$U \mid (d \bullet c) = (U \mid d) \bullet (U \mid c)$$
$$(d \circ c) \mid V = (d \mid V) \bullet (c \mid V).$$

(4) State extension with the tensor unit I as the trivial state space is isomorphic to the original component through maps of components:

$$\begin{tikzcd} T((X \otimes I) \otimes B) \arrow[r, "T(\rho\otimes\mathrm{id})"] & T(X \otimes B) & T((I \otimes X) \otimes B) \arrow[l, "T(\lambda\otimes\mathrm{id})"', "\cong"] \\ (X \otimes I) \otimes A \arrow[u, "c|I"] \arrow[r, "\rho\otimes\mathrm{id}", "\cong"'] & X \otimes A \arrow[u, "c"] & (I \otimes X) \otimes A \arrow[u, "I|c"] \arrow[l, "\lambda\otimes\mathrm{id}"', "\cong"] \end{tikzcd}$$

(5) Repeated state extension is isomorphic to tensored state extension:

$$\begin{tikzcd} T((U \otimes (V \otimes X)) \otimes B) \arrow[r, "T(\alpha\otimes\mathrm{id})", "\cong"'] & T(((U \otimes V) \otimes X) \otimes B) \\ (U \otimes (V \otimes X)) \otimes A \arrow[u, "U|(V|c)"] \arrow[r, "\alpha\otimes\mathrm{id}", "\cong"'] & ((U \otimes V) \otimes X) \otimes A \arrow[u, "(U\otimes V)|c"'] \end{tikzcd}$$

$$\begin{tikzcd} T((X \otimes (U \otimes V)) \otimes B) \arrow[r, "T(\alpha\otimes\mathrm{id})", "\cong"'] & T(((X \otimes U) \otimes V) \otimes B) \\ (X \otimes (U \otimes V)) \otimes A \arrow[u, "c|(U\otimes V)"] \arrow[r, "\alpha\otimes\mathrm{id}", "\cong"'] & ((X \otimes U) \otimes V) \otimes A \arrow[u, "(c|U)|V"'] \end{tikzcd}$$

(6) Left and right extension can be exchanged through an isomorphism of components:

$$\begin{tikzcd} T((U \otimes (X \otimes V)) \otimes B) \arrow[r, "T(\alpha\otimes\mathrm{id})", "\cong"'] & T(((U \otimes X) \otimes V) \otimes B) \\ (U \otimes (X \otimes V)) \otimes A \arrow[u, "U|(c|V)"] \arrow[r, "\alpha\otimes\mathrm{id}", "\cong"'] & ((U \otimes X) \otimes V) \otimes A \arrow[u, "(U|c)|V"'] \end{tikzcd}$$

(7) For $f : U \to U'$ and $g : V \to V'$ we have

$$(U' \mid c) \circ ((f \otimes \mathrm{id}) \otimes \mathrm{id}) = T((f \otimes \mathrm{id}) \otimes \mathrm{id}) \circ (U \mid c)$$
$$(d \mid V') \circ ((\mathrm{id} \otimes g) \otimes \mathrm{id}) = T((\mathrm{id} \otimes g) \otimes \mathrm{id}) \circ (d \mid V).$$

For a map h in $\mathbb{C}$ between states we have

$$T((\mathrm{id} \otimes h) \otimes \mathrm{id}) \circ (U \mid c) = U \mid (T(h \otimes \mathrm{id}) \circ c)$$
$$T((h \otimes \mathrm{id}) \otimes \mathrm{id}) \circ (d \mid V) = (T(h \otimes \mathrm{id}) \circ d) \mid V$$
$$(U \mid c) \circ ((\mathrm{id} \otimes h) \otimes \mathrm{id}) = U \mid (c \circ (h \otimes \mathrm{id}))$$
$$(d \mid V) \circ ((h \otimes \mathrm{id}) \otimes \mathrm{id}) = (d \circ (h \otimes \mathrm{id})) \mid V.$$

Consequently, if $h\colon X \to Y$ is a homomorphism of coalgebraic components, the following diagrams commute:

$$\begin{tikzcd}
T((U \otimes X) \otimes B) \arrow[r, "T((f\otimes h)\otimes \mathrm{id})"] & T((U' \otimes Y) \otimes B) \\
(U \otimes X) \otimes A \arrow[u, "U|c"] \arrow[r, "(f\otimes h)\otimes \mathrm{id}"] & (U' \otimes Y) \otimes A \arrow[u, "U'|d"']
\end{tikzcd}$$

$$\begin{tikzcd}
T((X \otimes V) \otimes B) \arrow[r, "T((h\otimes g)\otimes \mathrm{id})"] & T((Y \otimes V') \otimes B) \\
(X \otimes V) \otimes A \arrow[u, "c|V"] \arrow[r, "(h\otimes g)\otimes \mathrm{id}"] & (Y \otimes V') \otimes A \arrow[u, "d|V'"']
\end{tikzcd}$$

(8) State extension also commutes with relabelling: for $g\colon C \to A$ and $h\colon B \to D$ we have

$$(g,h)^*(U \mid c) = U \mid ((g,h)^*(c))$$
$$(g,h)^*(c \mid V) = ((g,h)^*(c)) \mid V.$$

Proof. Because left and right state extension are related through

$$c \mid V = T(\gamma \otimes \mathrm{id}) \circ (V \mid c) \circ (\gamma \otimes \mathrm{id}),$$

we generally only consider one case. For instance, for the first point, we have

$$\begin{aligned}
U \mid \eta &= T(\alpha) \circ \mathsf{st} \circ (\eta \otimes \mathrm{id}) \circ \alpha^{-1} \\
&= T(\alpha) \circ \mathsf{dst} \circ (\eta \otimes \eta) \circ \alpha^{-1} \\
&= T(\alpha) \circ \eta \circ \alpha^{-1} \\
&= \eta \circ \alpha \circ \alpha^{-1} \\
&= \eta.
\end{aligned}$$

The interaction of state extension and Kleisli composition in the third point of the lemma is best shown in the Kleisli category itself:

$$\begin{aligned}
(U \mid d) \bullet (U \mid c) &= \alpha \bullet (U \otimes d) \bullet \alpha^{-1} \bullet \alpha \bullet (U \otimes c) \bullet \alpha \\
&= \alpha \bullet (U \otimes d) \bullet (U \otimes c) \bullet \alpha \\
&= \alpha \bullet (U \otimes (d \bullet c)) \bullet \alpha \\
&= U \mid (d \bullet c).
\end{aligned}$$

The validity of the fourth point follows from

$$
\begin{tikzcd}
(X \otimes I) \otimes A \arrow[r, "\rho\otimes\mathrm{id}"] \arrow[d, "\gamma\otimes\mathrm{id}"'] \arrow[dddrrr, bend right=60, "c|I"'] & X \otimes A \arrow[rr, "c"] & & T(X \otimes B) \\
(I \otimes X) \otimes A \arrow[ur, "\lambda\otimes\mathrm{id}"] \arrow[d, "\alpha^{-1}"'] & & & \\
I \otimes (X \otimes A) \arrow[uur, "\lambda"] \arrow[d, "\mathrm{id}\otimes c"'] & & & \\
I \otimes T(X \otimes B) \arrow[r, "\mathsf{st}"'] \arrow[uuurrr, "\lambda"] & T(I \otimes (X \otimes B)) \arrow[r, "T(\alpha)"'] \arrow[uuurr, "T(\lambda)"] & T((I \otimes X) \otimes B) \arrow[r, "T(\gamma\otimes\mathrm{id})"'] \arrow[uuur, "T(\lambda\otimes\mathrm{id})"] & T((X \otimes I) \otimes B) \arrow[uuu, "T(\rho\otimes\mathrm{id})"']
\end{tikzcd}
$$

Also, for the fifth point, we will only elaborate one case, and in the Kleisli category:

$$
\begin{aligned}
(\alpha \otimes \mathrm{id}) \bullet (U \mid (V \mid c)) &= (\alpha \otimes \mathrm{id}) \bullet \alpha \bullet (U \otimes (\alpha \bullet (V \otimes c) \bullet \alpha^{-1})) \bullet \alpha^{-1} \\
&= \alpha \bullet \alpha \bullet (U \otimes (V \otimes c)) \bullet (U \otimes \alpha^{-1}) \bullet \alpha^{-1} \\
&= \alpha \bullet ((U \otimes V) \mid c) \bullet \alpha \bullet (U \otimes \alpha^{-1}) \bullet \alpha^{-1} \\
&= \alpha \bullet ((U \otimes V) \mid c) \bullet \alpha^{-1} \bullet (\alpha \otimes \mathrm{id}) \\
&= ((U \otimes V) \mid c) \bullet (\alpha \otimes \mathrm{id}).
\end{aligned}
$$

The verification of the remaining properties proceeds along the same lines, and is left as an exercise. □

3.1. *State extension as action*

The state extension operators | can be described as functors in the following diagram:

$$
\begin{tikzcd}
\mathbb{C} \times \mathbf{Comp}(T, A, B) \arrow[r, "|"] \arrow[d] & \mathbf{Comp}(T, A, B) \arrow[d] & \mathbf{Comp}(T, A, B) \times \mathbb{C} \arrow[l, "|"'] \arrow[d] \\
\mathbb{C} \times \mathbb{C} \arrow[r, "\otimes"] & \mathbb{C} & \mathbb{C} \times \mathbb{C} \arrow[l, "\otimes"']
\end{tikzcd}
$$

The functors

$$
\begin{aligned}
\mathbb{C} \times \mathbf{Comp}(T, A, B) &\to \mathbf{Comp}(T, A, B) \\
\mathbf{Comp}(T, A, B) \times \mathbb{C} &\to \mathbf{Comp}(T, A, B)
\end{aligned}
$$

at the top this diagram can be equivalently described as (two) functors of the form

$$\mathbb{C} \rightrightarrows [\mathbf{Comp}(T, A, B), \mathbf{Comp}(T, A, B)],$$

from $\mathbb{C}$ to the category of endofunctors on the category $\mathbf{Comp}(T, A, B)$. These two functors are strong monoidal, where the category of endofunctors carries composition and identity (of functors) as a monoidal structure. This means precisely that the monoidal category $\mathbb{C}$ acts on $\mathbf{Comp}(T, A, B)$ (see Janelidze and Kelly (2001)) in two ways, namely through left and right state extension.

Moreover, because (left and right) state extensions commute with relabelling, they form natural transformations in

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[rr, bend left=30, "\mathbb{C}\times\mathbf{Comp}(T,-,-)"] \arrow[rr, "\mathbf{Comp}(T,-,-)" description] \arrow[rr, bend right=30, "\mathbf{Comp}(T,-,-)\times\mathbb{C}"'] & \arrow[phantom, u, "\Downarrow"] \arrow[phantom, d, "\Uparrow"] & \mathbf{Cat} \end{tikzcd}$$

By combining these two observations, we conclude that the monoidal category $\mathbb{C}$ acts on the indexed category

$$\mathbf{Comp}(T,-,-) : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Cat},$$

in the sense that there are strong monoidal (left and right) state extension functors

$$\mathbb{C} \overset{\mathsf{lse}}{\underset{\mathsf{rse}}{\rightrightarrows}} \big[\mathbf{Comp}(T,-,-), \mathbf{Comp}(T,-,-)\big] \tag{6}$$

from $\mathbb{C}$ to the category of endomaps on $\mathbf{Comp}(T,-,-)$. Objects of the latter category are natural transformations

$$\sigma : \mathbf{Comp}(T,-,-) \Rightarrow \mathbf{Comp}(T,-,-)$$

and a morphism $M : \sigma \to \tau$ between such natural transformations is a so-called modification (Kelly and Street 1974; Jacobs 1999), which is a family of natural transformations $M_{(A,B)} : \sigma_{(A,B)} \to \tau_{(A,B)}$ commuting with relabelling.

We shall now elaborate the left state extension case in (6). Each $U \in \mathbb{C}$ yields a natural transformation

$$\mathsf{lse}(U) : \mathbf{Comp}(T,-,-) \to \mathbf{Comp}(T,-,-)$$

between indexed categories, with component on $(A,B) \in \mathbb{C}^{\mathrm{op}} \times \mathbb{C}$ given by the functor

$$\mathsf{lse}(U)_{(A,B)} : \mathbf{Comp}(T,A,B) \to \mathbf{Comp}(T,A,B),$$

which was written earlier as $c \mapsto U \mid c$. Hence $\mathsf{lse}(U)_{A,B} = (U \mid -)$. This is functorial by Lemma 3.2 (7), and natural in A, B by Lemma 3.2 (8). Each map $f : U \to U'$ in $\mathbb{C}$ yields a modification $\mathsf{lse}(f) : \mathsf{lse}(U) \to \mathsf{lse}(U')$ between these natural transformations, consisting of a family

$$\mathsf{lse}(f)_{(A,B)} : \mathsf{lse}(U)_{(A,B)} \to \mathsf{lse}(U')_{(A,B)}$$

of natural transformations, with component for a coalgebra $X \otimes A \xrightarrow{c} T(X \otimes B)$ consisting of the map

$$\mathsf{lse}(U)_{(A,B)}(c) = (U \mid c) \xrightarrow[= f \otimes \mathrm{id}_X]{\mathsf{lse}(f)_{(A,B),c}} (U' \mid c) = \mathsf{lse}(U')_{(A,B)}(c).$$

We thus have the natural transformations

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \arrow[d, bend right=50, "\mathbf{Comp}(T,-,-)"'] \arrow[d, bend left=50, "\mathbf{Comp}(T,-,-)"] \arrow[d, phantom, "\overset{\mathsf{lse}(U)}{\Longrightarrow}"] \\ \mathbf{Cat} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Comp}(T,A,B) \arrow[d, bend right=50, "\mathsf{lse}(U)_{(A,B)}"'] \arrow[d, bend left=50, "\mathsf{lse}(U')_{(A,B)}"] \arrow[d, phantom, "\overset{\mathsf{lse}(f)_{(A,B)}}{\Longrightarrow}"] \\ \mathbf{Comp}(T,A,B) \end{tikzcd}$$

It is not hard to check that $\mathsf{lse}(U)$ is natural (in A, B) and that these $\mathsf{lse}(f)$ are natural (in c) and commute with relabelling.

Finally, this functor

$$\mathsf{lse} : \mathbb{C} \to [\mathbf{Comp}(T, -, -), \mathbf{Comp}(T, -, -)]$$

is strong monoidal because it preserves the monoidal structure:

$$\begin{aligned}
\mathsf{lse}(I)_{(A,B)}(c) &= I \mid c \\
&\cong c \\
\mathsf{lse}(U \otimes V)_{(A,B)}(c) &= (U \otimes V) \mid c \\
&\cong U \mid (V \mid c) \\
&= \mathsf{lse}(U)_{(A,B)}\big(\mathsf{lse}(V)_{(A,B)}(c)\big) \\
&= \big(\mathsf{lse}(U)_{(A,B)} \circ \mathsf{lse}(V)_{(A,B)}\big)(c),
\end{aligned}$$

where the category of endomaps $[\mathbf{Comp}(T, -, -), \mathbf{Comp}(T, -, -)]$ carries the standard monoidal structure given by composition as tensor, with identity as the tensor unit.

4. Composition of components

This section describes three basic forms of composition for coalgebraic components, namely:

— sequential composition $\ggg$
— multiplicative parallel composition $\|$
— additive parallel composition $\square$.

We will begin with $\ggg$, which we can conveniently describe in terms of state extension.

Definition 4.1. The sequential composition operator $\ggg$ is defined for coalgebraic components with matching input and output. For

$$X \otimes A \xrightarrow{c} T(X \otimes B) \quad \text{and} \quad Y \otimes B \xrightarrow{d} T(Y \otimes C)$$

we get $c \ggg d$ through the composition of Kleisli maps:

$$c \ggg d = \Big((X \otimes Y) \otimes A \xrightarrow{c|Y} (X \otimes Y) \otimes B \xrightarrow{X|d} (X \otimes Y) \otimes C\Big).$$

Thus $c \ggg d$ involves first doing c and then d, on a combined state space $X \otimes Y$.

The notation $\ggg$ for composition is as used for *arrows* – see, for example, Jacobs *et al.* (2009). Composition of components satisfies the properties of composition for arrows, but only up to (canonical) isomorphisms. This will be shown next.

Lemma 4.2. The following equations and isomorphisms (of coalgebraic components) hold for sequential composition.

(1) We have

$$\mathsf{arr}(f) \ggg \mathsf{arr}(g) = (\mathsf{arr}(g \circ f) \mid I) = (I \mid \mathsf{arr}(g \circ f)) \xrightarrow[\cong]{\lambda=\rho} \mathsf{arr}(g \circ f)$$

in the following isomorphism of coalgebraic components.

$$\begin{tikzcd} T((I \otimes I) \otimes C) \arrow[r, "T(\lambda\otimes\mathrm{id})=T(\rho\otimes\mathrm{id})", "\cong"'] & T(I \otimes C) \\ (I \otimes I) \otimes A \arrow[u, "\mathsf{arr}(f)\ggg\mathsf{arr}(g)=(\mathsf{arr}(g\circ f)|I)=(I|\mathsf{arr}(g\circ f))"] \arrow[r, "\cong", "(\lambda\otimes\mathrm{id})=(\rho\otimes\mathrm{id})"'] & I \otimes A \arrow[u, "\mathsf{arr}(g\circ f)"'] \end{tikzcd}$$

(2) We have

$$c \ggg \mathsf{arr}(g) \xrightarrow[\cong]{\rho} T(\mathrm{id} \otimes g) \circ c$$

$$\mathsf{arr}(f) \ggg d \xrightarrow[\cong]{\lambda} d \circ (\mathrm{id} \otimes f)$$

in

$$\begin{tikzcd} T((X \otimes I) \otimes C) \arrow[r, "T(\rho\otimes\mathrm{id})", "\cong"'] & T(X \otimes C) \\ (X \otimes I) \otimes A \arrow[u, "c\ggg\mathsf{arr}(g)"] \arrow[r, "\cong", "\rho\otimes\mathrm{id}"'] & X \otimes A \arrow[u, "T(\mathrm{id}\otimes g)\circ c"'] \end{tikzcd} \qquad \begin{tikzcd} T((I \otimes Y) \otimes C) \arrow[r, "T(\lambda\otimes\mathrm{id})", "\cong"'] & T(Y \otimes C) \\ (I \otimes Y) \otimes A \arrow[u, "\mathsf{arr}(f)\ggg d"] \arrow[r, "\cong", "\lambda\otimes\mathrm{id}"'] & Y \otimes A \arrow[u, "d\circ(\mathrm{id}\otimes f)"'] \end{tikzcd}$$

In particular, $\mathsf{arr}(\mathrm{id})$ is unit for $\ggg$, up-to-isomorphism.

(3) We have

$$(c \ggg (d \ggg e)) \xrightarrow[\cong]{\alpha} ((c \ggg d) \ggg e)$$

in

$$\begin{tikzcd} T((X \otimes (Y \otimes Z)) \otimes D) \arrow[r, "T(\alpha\otimes\mathrm{id})", "\cong"'] & T(((X \otimes Y) \otimes Z) \otimes D) \\ (X \otimes (Y \otimes Z)) \otimes A \arrow[u, "(c\ggg(d\ggg e))"] \arrow[r, "\cong", "\alpha\otimes\mathrm{id}"'] & ((X \otimes Y) \otimes Z) \otimes A \arrow[u, "((c\ggg d)\ggg e)"'] \end{tikzcd}$$

(4) For appropriately typed maps in $\mathbb{C}$ between states,

$$((f \otimes g) \otimes \mathrm{id}) \bullet (c \ggg d) = ((f \otimes \mathrm{id}) \bullet c) \ggg ((g \otimes \mathrm{id}) \bullet d)$$
$$(c \ggg d) \bullet ((f \otimes g) \otimes \mathrm{id}) = (c \bullet (f \otimes \mathrm{id})) \ggg (d \bullet (g \otimes \mathrm{id}))$$

As a result, sequential composition $\ggg$ of components is a functor of the form

$$\mathbf{Comp}(T, A, B) \times \mathbf{Comp}(T, B, C) \to \mathbf{Comp}(T, A, C).$$

The last point suggests the notation used in Hasuo *et al.* (2009) for the type of $\ggg$, namely, $(A, B) \times (B, C) \to (A, C)$. We shall also use it later on, especially in Section 6.

Proof. All these properties follow from Lemma 3.2. The numbers labelling the equations below refer to the items in this lemma. Recall that we use $\bullet$ for composition in the Kleisli category of the monad T and $\circ$ for composition in $\mathbb{C}$.

(1) We have,

$$\begin{aligned}
\mathsf{arr}(f) \ggg \mathsf{arr}(g) &= (\mathsf{arr}(g) \mid I) \bullet (I \mid \mathsf{arr}(f)) \\
&\overset{(2)}{=} \mu \circ T(\eta \circ (\mathrm{id} \otimes g)) \circ \eta \circ (\mathrm{id} \otimes f) \\
&= T(\mathrm{id} \otimes g) \circ \eta \circ (\mathrm{id} \otimes f) \\
&= \eta \circ (\mathrm{id} \otimes g) \circ (\mathrm{id} \otimes f) \\
&= \eta \circ (\mathrm{id} \otimes (g \circ f)) \\
&\overset{(2)}{=} \mathsf{arr}(g \circ f) \mid I \\
&= I \mid \mathsf{arr}(g \circ f) \\
&\xrightarrow[\cong]{\lambda} \mathsf{arr}(g \circ f).
\end{aligned}$$

(2) Similarly,

$$\begin{aligned}
T(\rho \otimes \mathrm{id}) \circ (c \ggg \mathsf{arr}(g)) &= T(\rho \otimes \mathrm{id}) \circ ((X \mid \mathsf{arr}(g)) \bullet (c \mid I)) \\
&\overset{(2)}{=} T(\rho \otimes \mathrm{id}) \circ \mu \circ T(\eta \circ (\mathrm{id} \otimes g)) \circ (c \mid I) \\
&= T(\rho \otimes \mathrm{id}) \circ T(\mathrm{id} \otimes g) \circ (c \mid I) \\
&= T(\mathrm{id} \otimes g) \circ T(\rho \otimes \mathrm{id}) \circ (c \mid I) \\
&\overset{(4)}{=} T(\mathrm{id} \otimes g) \circ (c \mid I) \circ (\rho \otimes \mathrm{id}).
\end{aligned}$$

(3) Associativity of $\ggg$ follows from a straightforward calculation, which is best done in the Kleisli category:

$$\begin{aligned}
&(\alpha \otimes \mathrm{id}) \bullet \big(c \ggg (d \ggg e)\big) \\
&\quad= (\alpha \otimes \mathrm{id}) \bullet (X \mid d \ggg e) \bullet (c \mid Y \otimes Z) \\
&\quad= (\alpha \otimes \mathrm{id}) \bullet \big(X \mid ((Y \mid e) \bullet (d \mid Z))\big) \bullet (c \mid Y \otimes Z) \\
&\quad\overset{(3)}{=} (\alpha \otimes \mathrm{id}) \bullet (X \mid (Y \mid e)) \bullet (X \mid (d \mid Z)) \bullet (c \mid Y \otimes Z) \\
&\quad\overset{(5)}{=} (X \otimes Y \mid e) \bullet (\alpha \otimes \mathrm{id}) \bullet (X \mid (d \mid Z)) \bullet (c \mid Y \otimes Z) \\
&\quad\overset{(6)}{=} (X \otimes Y \mid e) \bullet (X \mid d) \mid Z) \bullet (\alpha \otimes \mathrm{id}) \bullet (c \mid Y \otimes Z) \\
&\quad\overset{(5)}{=} (X \otimes Y \mid e) \bullet ((X \mid d) \mid Z) \bullet ((c \mid Y) \mid Z) \bullet (\alpha \otimes \mathrm{id}) \\
&\quad\overset{(3)}{=} (X \otimes Y \mid e) \bullet \big(((X \mid d) \bullet (c \mid Y)) \mid Z\big) \bullet (\alpha \otimes \mathrm{id}) \\
&\quad= (X \otimes Y \mid e) \bullet (c \ggg d \mid Z) \bullet (\alpha \otimes \mathrm{id}) \\
&\quad= \big((c \ggg d) \ggg e\big) \bullet (\alpha \otimes \mathrm{id}).
\end{aligned}$$

(4) We will only prove functoriality, in a direct way: for two maps of coalgebraic components $f : c_1 \to c_2$ and $g : d_1 \to d_2$, where $c_i : X_i {\otimes} A \to T(X_i {\otimes} B)$ and $d_i : Y_i {\otimes} B \to T(Y_i \otimes X)$ the map $f \otimes g : X_1 \otimes Y_1 \to X_2 \otimes Y_2$ is a morphism of composite coalgebraic

components:

$$\begin{aligned}((f \otimes g) \otimes \mathrm{id}_C) \bullet (c_1 \ggg d_1) &= ((f \otimes g) \otimes \mathrm{id}_C) \bullet (X_1 \mid d_1) \bullet (c_1 \mid Y_1) \\ &\overset{(7)}{=} (X_2 \mid d_2) \bullet ((f \otimes g) \otimes \mathrm{id}_B)) \bullet (c_1 \mid Y_1) \\ &\overset{(7)}{=} (X_2 \mid d_2) \bullet (c_2 \mid Y_2) \bullet ((f \otimes g) \otimes \mathrm{id}_A) \\ &= (c_2 \ggg d_2) \bullet ((f \otimes g) \otimes \mathrm{id}_A). \qquad \square\end{aligned}$$

The next result captures the interaction between sequential composition $\ggg$ and state extension $|$.

Lemma 4.3. For components $X \otimes A \xrightarrow{c} T(X \otimes B)$ to $Y \otimes B \xrightarrow{d} T(Y \otimes C)$, there are associativity isomorphisms:

$$\Big(U \mid (c \ggg d)\Big) \xrightarrow[\cong]{\alpha} \Big((U \mid c) \ggg d\Big)$$
$$\Big(c \ggg (d \mid V)\Big) \xrightarrow[\cong]{\alpha} \Big((c \ggg d) \mid V\Big).$$

Proof. We use the properties of Lemma 3.2. We will only consider the first associativity isomorphism:

$$\begin{aligned}(\alpha \otimes \mathrm{id}) \bullet \Big(U \mid (c \ggg d)\Big) &= (\alpha \otimes \mathrm{id}) \bullet \big(U \mid ((X \mid d) \bullet (c \mid Y))\big) \\ &\overset{(7)}{=} (\alpha \otimes \mathrm{id}) \bullet (U \mid (X \mid d)) \bullet (U \mid (c \mid Y)) \\ &\overset{(5)}{=} ((U \otimes X) \mid d) \bullet (\alpha \otimes \mathrm{id}) \bullet (U \mid (c \mid Y)) \\ &\overset{(6)}{=} ((U \otimes X) \mid d) \bullet ((U \mid c) \mid Y) \bullet (\alpha \otimes \mathrm{id}) \\ &= \Big((U \mid c) \ggg d\Big) \bullet (\alpha \otimes \mathrm{id}). \qquad \square\end{aligned}$$

4.1. *Multiplicative parallel composition*

Two coalgebraic components c, d, with different state spaces, and different inputs & outputs, can be put in parallel to form new components. This can be done in different ways. We will begin by discussing the 'multiplicative' method, which involves taking the tensor of the inputs & outputs. Later, we will describe the 'additive' parallel composition, which involves the coproduct of inputs & outputs. The additive version turns out to be more important in the current setting, so our discussion of the multiplicative version will be rather brief.

Definition 4.4. For components

$$X \otimes A \xrightarrow{c} T(X \otimes B)$$
$$Y \otimes C \xrightarrow{d} T(Y \otimes D)$$

the multiplicative parallel composition $c \parallel d$ is defined as Kleisli composition:

$$\begin{tikzcd}
(X \otimes Y) \otimes (A \otimes C) \arrow[r, "c\|d"] \arrow[d, "\widehat{\gamma}"', "\cong"] & (X \otimes Y) \otimes (B \otimes D) \\
(X \otimes A) \otimes (Y \otimes C) \arrow[r, "c\otimes d"] & (X \otimes B) \otimes (Y \otimes D) \arrow[u, "\cong", "\widehat{\gamma}"']
\end{tikzcd}$$

where $\widehat{\gamma}$ is the obvious isomorphism that swaps the inner two objects.

It is easy to see that $\parallel$ yields a functor:

$$\parallel \quad : \quad \mathbf{Comp}(T, A, B) \times \mathbf{Comp}(T, C, D) \longrightarrow \mathbf{Comp}(T, A \otimes C, B \otimes D). \tag{7}$$

Using the (multi-sorted) Lawvere theory notation of Hasuo *et al.* (2009), this operator can be described as a map

$$\parallel : (A, B) \times (C, D) \to (A \otimes C, B \otimes D).$$

Remark 4.5. The type (7) of the functor $\parallel$ even suggests that the correspondence $\mathbf{Comp}(T, -, -)$ be a (lax) monoidal functor $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Cat}$, where the former has an obvious monoidal structure (inherited from $\mathbb{C}$) and the latter has Cartesian products as tensor products. This is also true of the additive parallel composition functor $\Box$ studied in Section 4.2. However, it is not yet clear how we should use such higher-dimensional structures, so the relevant technical developments are left for future work.

Now that we have this $\parallel$ operator, we can describe the equivalents of the 'first' and 'second' operators in the context of Hughes' Arrows (Hughes 2000). They add an additional input & output, on the left or on the right of the existing input & output, namely, through

$$\begin{aligned} \mathsf{first}_{\parallel}(c) &= c \parallel \mathsf{arr}(\mathrm{id}) \\ \mathsf{second}_{\parallel}(c) &= \mathsf{arr}(\mathrm{id}) \parallel c. \end{aligned} \tag{8}$$

We will only state the following result for $\parallel$ without proof.

Lemma 4.6. There are isomorphisms of components:

$$\Big(U \mid (c \parallel d)\Big) \xrightarrow[\cong]{\alpha} \Big((U \mid c) \parallel d\Big)$$

$$\Big(c \parallel (d \mid V)\Big) \xrightarrow[\cong]{\alpha} \Big((c \parallel d) \mid V\Big). \qquad \Box$$

4.2. *Additive parallel composition*

Our next goal is to define an additive parallel composition operator $\Box$ for coalgebraic components, which is called 'external choice' in Barbosa (2001; 2003). We need to assume that our category $\mathbb{C}$ has binary coproducts $+$, and that the tensor $\otimes$ distributes over them, as described in Section 2, through a distribution map dis as in (2). These coproducts $+$ in $\mathbb{C}$ also form coproducts in the Kleisli category $\mathcal{K}\ell(T)$ and are preserved by $J : \mathbb{C} \to \mathcal{K}\ell(T)$.

So in $\mathcal{K}\ell(T)$ we have coprojections

$$J(\kappa_i) = \eta \circ \kappa_i : X_i \to T(X_1 + X_2)$$

with cotupling as in $\mathbb{C}$. Since the monad T is assumed to be commutative (that is, symmetric monoidal), $\otimes$ is also a tensor in $\mathcal{K}\ell(T)$, and it distributes over $+$ in $\mathcal{K}\ell(T)$, through $J(\mathsf{dis}) = \eta \circ \mathsf{dis}$ as distributivity isomorphism.

Definition 4.7. An additive parallel operator $\square$ is defined on coalgebraic components

$$X \otimes A \xrightarrow{c} T(X \otimes B)$$

$$Y \otimes C \xrightarrow{d} T(Y \otimes D)$$

as Kleisli composition:

$$\begin{tikzcd}
(X \otimes Y) \otimes (A + C) \arrow[r, "c \square d"] \arrow[d, "\mathsf{dis}^{-1}"', "\cong"] & (X \otimes Y) \otimes (B + D) \\
(X \otimes Y) \otimes A + (X \otimes Y) \otimes C \arrow[r, "c|Y + X|d"] & (X \otimes Y) \otimes B + (X \otimes Y) \otimes D \arrow[u, "\cong", "\mathsf{dis}"']
\end{tikzcd}$$

where $c \mid Y + X \mid d$ is the coproduct of maps in the Kleisli category.

This additive composition operator $\square$ forms a functor

$$\mathbf{Comp}(T, A, B) \times \mathbf{Comp}(T, C, D) \to \mathbf{Comp}(T, A + C, B + D),$$

which is defined on morphisms by $f \square g = f \otimes g$. It may thus be written as a map

$$\square : (A, B) \times (C, D) \to (A + C, B + D).$$

This will be helpful in Section 6.

Just as we had 'first' and 'second' operators for multiplicative parallel composition (8), we also have them in the additive case:

$$\begin{aligned} \mathsf{first}_{\square}(c) &= c \square \mathsf{arr}(\mathrm{id}) \\ \mathsf{second}_{\square}(c) &= \mathsf{arr}(\mathrm{id}) \square c. \end{aligned} \tag{9}$$

These fundamental operators occur frequently in the rest of the paper, for instance in the (di)naturality properties of the trace operator in Section 6.

We make the relation between $\square$ and state extension explicit. It is very similar to the relations between $\ggg$ or $\parallel$ and state extension, see Lemmas 4.3 and 4.6.

Lemma 4.8. For components

$$X \otimes A \xrightarrow{c} T(X \otimes B)$$

to

$$Y \otimes C \xrightarrow{d} T(Y \otimes D)$$

there are associativity isomorphisms

$$\Big(U \mid (c \square d)\Big) \xrightarrow[\cong]{\alpha} \Big((U \mid c) \square d\Big)$$

$$\Big(c \square (d \mid V)\Big) \xrightarrow[\cong]{\alpha} \Big((c \square d) \mid V\Big).$$

Proof. The proof requires a rather elaborate calculation, and we will only present it for the first isomorphism:

$$
\begin{aligned}
&\big((U \mid c) \,\square\, d\big) \bullet (\alpha \otimes \mathrm{id}) \\
&\quad = \quad \mathsf{dis} \bullet \big(((U \mid c) \mid Y) + ((U \otimes X) \mid d)\big) \bullet \mathsf{dis}^{-1} \bullet (\alpha \otimes \mathrm{id}) \\
&\quad \overset{2.1\,(1)}{=} \quad \mathsf{dis} \bullet \big(((U \mid c) \mid Y) + ((U \otimes X) \mid d)\big) \bullet ((\alpha \otimes \mathrm{id}) + (\alpha \otimes \mathrm{id})) \bullet \mathsf{dis}^{-1} \\
&\quad \overset{2.1\,(2)}{=} \quad \mathsf{dis} \bullet \big((((U \mid c) \mid Y) \bullet (\alpha \otimes \mathrm{id})) + (((U \otimes X) \mid d) \bullet (\alpha \otimes \mathrm{id}))\big) \bullet \\
&\qquad\qquad\qquad (\alpha + \alpha) \bullet \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet \alpha^{-1} \\
&\quad \overset{3.2\,(6),(5)}{=} \quad \mathsf{dis} \bullet \big(((\alpha \otimes \mathrm{id}) \bullet (U \mid (c \mid Y)) \bullet \alpha) + ((\alpha \otimes \mathrm{id}) \bullet (U \mid (X \mid d)) \bullet \alpha)\big) \bullet \\
&\qquad\qquad\qquad \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet \alpha^{-1} \\
&\quad = \quad \mathsf{dis} \bullet \big((\alpha \otimes \mathrm{id}) + (\alpha \otimes \mathrm{id})\big) \bullet \\
&\qquad\qquad\qquad \big((\alpha \bullet (U \otimes (c \mid Y)) \bullet \alpha^{-1} \bullet \alpha) + (\alpha \bullet (U \otimes (X \mid d)) \bullet \alpha^{-1} \bullet \alpha)\big) \bullet \\
&\qquad\qquad\qquad\quad \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet \alpha^{-1} \\
&\quad \overset{2.1\,(1)}{=} \quad (\alpha \otimes \mathrm{id}) \bullet \mathsf{dis} \bullet (\alpha + \alpha) \bullet \big((U \otimes (c \mid Y)) + (U \otimes (X \mid d))\big) \bullet \\
&\qquad\qquad\qquad \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet \alpha^{-1} \\
&\quad \overset{2.1\,(2)}{=} \quad (\alpha \otimes \mathrm{id}) \bullet \alpha \bullet (U \otimes \mathsf{dis}) \bullet \mathsf{dis} \bullet \big((U \otimes (c \mid Y)) + (U \otimes (X \mid d))\big) \bullet \\
&\qquad\qquad\qquad \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet \alpha^{-1} \\
&\quad \overset{2.1\,(1)}{=} \quad (\alpha \otimes \mathrm{id}) \bullet \alpha \bullet (U \otimes \mathsf{dis}) \bullet (U \otimes (c \mid Y + X \mid d)) \bullet (U \otimes \mathsf{dis}^{-1}) \bullet \alpha^{-1} \\
&\quad = \quad (\alpha \otimes \mathrm{id}) \bullet \alpha \bullet (U \otimes (c \,\square\, d)) \bullet \alpha^{-1} \\
&\quad = \quad (\alpha \otimes \mathrm{id}) \bullet \big(U \mid (c \,\square\, d)\big). \qquad \square
\end{aligned}
$$

Many other properties, like associativity, can be proved for $\square$. We will state some of them explicitly, but leave the details of the (straightforward) proofs as an exercise.

Lemma 4.9.

(1) Given arrows $f : A \to B$ and $g : C \to D$ in $\mathbb{C}$, we have a canonical isomorphism of coalgebraic components $\mathsf{arr}(f) \,\square\, \mathsf{arr}(g) \xrightarrow{\cong} \mathsf{arr}(f + g)$. That is,

$$
\begin{tikzcd}
T((I \otimes I) \otimes (B + D)) \arrow[r, "T(\lambda\otimes\mathrm{id})=T(\rho\otimes\mathrm{id})", "\cong"'] & T(I \otimes (B + D)) \\
(I \otimes I) \otimes (A + C) \arrow[u, "\mathsf{arr}(f)\square\mathsf{arr}(g)"] \arrow[r, "\lambda\otimes\mathrm{id}=\rho\otimes\mathrm{id}", "\cong"'] & I \otimes (A + C) \arrow[u, "\mathsf{arr}(f+g)"']
\end{tikzcd}
$$

(2) We have

$$
(c \ggg c') \,\square\, (d \ggg d') \xrightarrow{\cong} (c \,\square\, d) \ggg (c' \,\square\, d'),
$$

in

$$\begin{tikzcd} T((X \otimes X') \otimes (Y \otimes Y') \otimes (A'' + B'')) \arrow[r, "T(\beta \otimes \mathrm{id})", "\cong"'] & T((X \otimes Y) \otimes (X' \otimes Y') \otimes (A'' + B'')) \\ (X \otimes X') \otimes (Y \otimes Y') \otimes (A + B) \arrow[u, "(c \ggg c') \Box (d \ggg d')"] \arrow[r, "\beta \otimes \mathrm{id}", "\cong"'] & (X \otimes Y) \otimes (X' \otimes Y') \otimes (A + B) \arrow[u, "(c \Box d) \ggg (c' \Box d')"'] \end{tikzcd}$$

Here

$$\beta : (X \otimes X') \otimes (Y \otimes Y') \xrightarrow{\cong} (X \otimes Y) \otimes (X' \otimes Y')$$

is the canonical isomorphism in a symmetric monoidal category $(\mathbb{C}, \otimes, I)$.

(3) We have

$$(c \Box (d \Box e)) \ggg \mathsf{arr}(\alpha_+) \xrightarrow{\cong} \mathsf{arr}(\alpha_+) \ggg ((c \Box d) \Box e),$$

in

$$\begin{tikzcd} T(((X \otimes (Y \otimes Z)) \otimes I) \otimes ((D + E) + F)) \arrow[r, "T(\beta' \otimes \mathrm{id})", "\cong"'] & T((I \otimes ((X \otimes Y) \otimes Z)) \otimes ((D + E) + F)) \\ ((X \otimes (Y \otimes Z)) \otimes I) \otimes (A + (B + C)) \arrow[u, "(c \Box (d \Box e)) \ggg \mathsf{arr}(\alpha_+)"] \arrow[r, "\beta' \otimes \mathrm{id}", "\cong"'] & (I \otimes ((X \otimes Y) \otimes Z)) \otimes (A + (B + C)) \arrow[u, "\mathsf{arr}(\alpha_+) \ggg ((c \Box d) \Box e)"'] \end{tikzcd}$$

Here

$$\beta' : (X \otimes (Y \otimes Z)) \otimes I \xrightarrow{\cong} I \otimes ((X \otimes Y) \otimes Z)$$

is the canonical isomorphism in $\mathbb{C}$, and α_+ is the associativity isomorphism for $+$.

(4) We have

$$(c \Box d) \ggg \mathsf{arr}(\gamma_+) \xrightarrow{\cong} \mathsf{arr}(\gamma_+) \ggg (d \Box c),$$

in

$$\begin{tikzcd} T(((X \otimes Y) \otimes I) \otimes (D + C)) \arrow[r, "T(\beta'' \otimes \mathrm{id})", "\cong"'] & T((I \otimes (Y \otimes X)) \otimes (D + C)) \\ ((X \otimes Y) \otimes I) \otimes (A + B) \arrow[u, "(c \Box d) \ggg \mathsf{arr}(\gamma_+)"] \arrow[r, "\beta'' \otimes \mathrm{id}", "\cong"'] & (I \otimes (Y \otimes X)) \otimes (A + B) \arrow[u, "\mathsf{arr}(\gamma_+) \ggg (d \Box c)"'] \end{tikzcd}$$

Here

$$\beta'' = \lambda^{-1} \circ \gamma \circ \rho : (X \otimes Y) \otimes I \xrightarrow{\cong} I \otimes (Y \otimes X)$$

and γ_+ is the symmetry isomorphism for $+$.

Proof. Items (1)–(2) are easy by direct calculation. Items (3)–(4) follow, essentially, from the naturality of α_+ and γ_+, and Lemma 2.2 (2)–(3). $\Box$

4.3. *Method combination*

The additive parallel composition $c \,\Box\, d$ from the previous subsection applies to arbitrary components c, d, which typically have different state spaces. In the special case when c, d share the same state space, there is also a composition operator, which we shall write as $\{c, d\}$. This can be viewed as a combination of the (Java-like) methods of c and d on their shared state space.

Definition 4.10. For two components

$$X \otimes A \xrightarrow{c} T(X \otimes B)$$

to

$$X \otimes C \xrightarrow{d} T(X \otimes D)$$

with the same state space X, we define $\{c, d\}$ as Kleisli composition:

$$\begin{tikzcd} X \otimes (A + C) \arrow[r, "\{c,d\}"] \arrow[d, "\mathsf{dis}^{-1}"', "\cong"] & X \otimes (B + D) \\ X \otimes A + X \otimes C \arrow[r, "c+d"] & X \otimes B + X \otimes D \arrow[u, "\mathsf{dis}"', "\cong"] \end{tikzcd}$$

If we understand a coalgebraic component as a mathematical model of a class in an object-oriented programming language, we can see this method combination operator as a form of class building: first the state space X is fixed, and then methods

$$c_i : X \otimes A_i \to T(X \otimes B_i)$$

are combined in a class

$$c = \{c_1, \ldots, c_n\} : X \otimes (A_1 + \cdots + A_n) \to T(X \otimes (B_1 + \cdots + B_n)).$$

Moreover, the extension of classes can be described to give a form of subclass and inheritance, albeit without the overriding of methods. Given a class/component

$$c : X \otimes A \to T(X \otimes B),$$

we can form a subclass by first extending the state to

$$c \mid Y : (X \otimes Y) \otimes A \to T((X \otimes Y) \otimes B).$$

Indeed, subclassing involves an extended state to accommodate additional fields/attributes. Additional methods

$$d : (X \otimes Y) \otimes C \to T((X \otimes Y) \otimes D)$$

may now be added to obtain a subclass

$$c' = \{c \mid Y, d\} : (X \otimes Y) \otimes (A + C) \to T((X \otimes Y) \otimes (B + D)).$$

The proof of the following result is left as an exercise.

Lemma 4.11. Method combination commutes with state extension:

$$U \mid \{c, d\} = \{U \mid c, U \mid d\}$$
$$\{c, d\} \mid V = \{c \mid V, d \mid V\}.$$

5. Tube diagrams for components

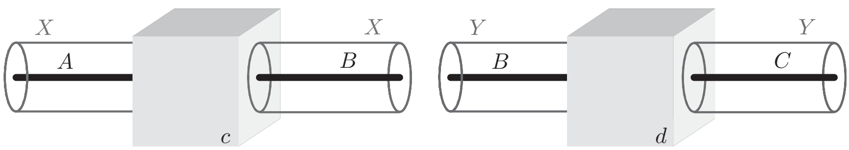
Our goal in Section 6 (which is central in this paper) will be first to introduce a trace operator that realises feedback loops for components and then prove that this operator does indeed satisfy the expected equational properties from Joyal *et al.* (1996), such as dinaturality, yanking and superposing. It turns out, however, that the composed components occurring in the equations are rather complicated, and their overall structures are best described using a variant of *string diagrams*.

String diagrams were introduced in Penrose (1971) to provide a succinct representation of the morphisms in a monoidal category – see also Joyal and Street (1991). Our problem is that we need to deal with two different kinds of monoidal structure $\otimes$ and $+$, and this calls for a carefully devised pictorial convention. Following McCurdy (2010), we employ a version of string diagrams augmented by tubes, which we call *tube diagrams*. Tubes enhance the slightly more common pictorial convention of *functorial boxes* (Cockett and Seely 1999; Melliès 2006). In the current paper, tubes capture applications of functors of the form $X \otimes -$.

We will now present the tube diagrams (based on McCurdy (2010)) for some of the composition operators introduced in Section 4. We will begin by drawing some diagrams, and then give an explanation. First consider the sequential composition operator in Definition 4.1. Given two components

$$X \otimes A \xrightarrow{c} X \otimes B$$
$$Y \otimes B \xrightarrow{d} Y \otimes C$$

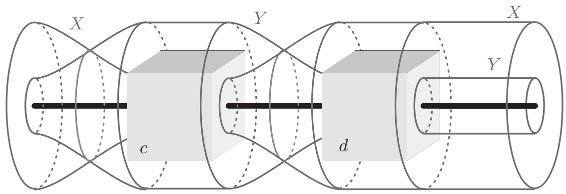
in $\mathscr{K}\ell(T)$ with matching input and output, we represent them using tube diagrams as follows:



Their composition

$$(X \otimes Y) \otimes A \xrightarrow{c \ggg d} (X \otimes Y) \otimes C$$

is then depicted as follows:

$$c \ggg d \quad =$$ 

(10)

We use the following conventions for tube diagrams:

— Diagrams are read from left to right.

— Each tube designates an object in $\mathscr{K}\ell(T)$. More precisely, it designates the identity morphism on the object. In diagram (10), the tube that is shrunk and plugged into the c-box and the tube that comes out of c and is expanded to enclose the d-box are both of type X. The other two tubes, *viz.* the one that encloses c and the one that comes out of d, are both of type Y. The three thick lines through the centre are of type A, B and C. In fact, these 'lines' are also tubes and are just shown as lines here to simplify the picture.

— Nested tubes designate the tensor $\otimes$ in $\mathscr{K}\ell(T)$, and are calculated from the outermost to the innermost. For example, the collection of three nested tubes at the left-hand end of diagram (10) represents the object $X \otimes Y \otimes A$.

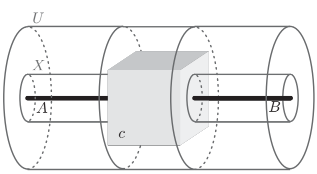
— Symmetry γ of the tensor $\otimes$ is depicted as a 'waist', that is, the exchange of outer and inner tubes. This occurs twice in diagram (10), where the 'waists' are marked with circles.

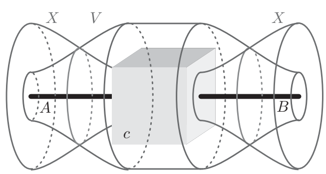
— Associativity isomorphisms are left implicit. That is, when dealing with tube diagrams, we will assume *strict* monoidal structures in which we do not distinguish $(X \otimes Y) \otimes A$ from $X \otimes (Y \otimes A)$, or $(X + Y) + A$ from $X + (Y + A)$. For this reason, our later use of tube diagrams should be viewed only as a 'guideline' for rigorous calculational proofs, rather than as proofs in themselves. We will return to this point later in Remark 5.2.

To summarise, diagram (10) can be 'parsed' into the following composition of morphisms in $\mathscr{K}\ell(T)$:

$$X \otimes Y \otimes A \xrightarrow[\text{waist}]{\gamma\otimes A} Y \otimes X \otimes A \xrightarrow[\text{box in a tube}]{Y\otimes c} Y \otimes X \otimes B \xrightarrow[\text{waist}]{\gamma\otimes B} X \otimes Y \otimes B \xrightarrow[\text{box in a tube}]{X\otimes d} X \otimes Y \otimes C$$

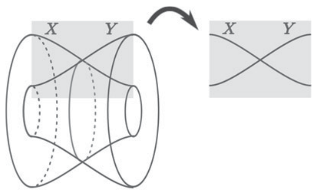
This composition is therefore the same thing as in Definition 4.1, modulo the use of the associativity isomorphisms α. It is implicit in this correspondence that left- and right-state extensions (Definition 3.1) can be depicted as follows:

$$U|c =$$ 

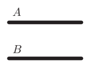
$$c|V =$$ 

(11)

The 'waist' diagram representing the symmetry $\gamma : X \otimes Y \stackrel{\cong}{\Rightarrow} Y \otimes X$ might seem strange at first sight, but, in fact, the usual 'crossing' representation of symmetry (see, for example, Joyal and Street (1991)) can be recovered by looking at a certain section of the three-dimensional picture†, as in:



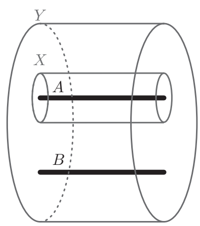
In contrast to the multiplicative tensor $\otimes$, the additive tensor $+$ is depicted by putting two tubes in parallel (rather than nested). For example,



for

$$A + B$$

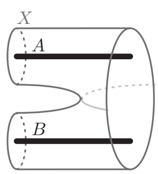
and



for

$$Y \otimes ((X \otimes A) + B).$$

The distributivity isomorphism dis from (2) relating the two tensors $\otimes$ and $+$ has a nice graphical representation as a pair of 'pants', so



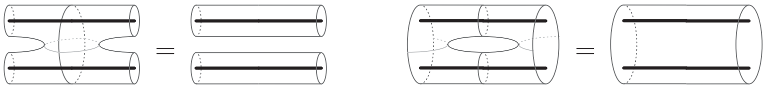
(12)

represents

$$\mathsf{dis} : X \otimes A + X \otimes B \longrightarrow X \otimes (A + B).$$

† This observation is due to Shin-ya Katsumata.

Furthermore, as dis is an isomorphism, the following equalities hold (the right-hand sides are identity maps on suitable objects):



(13)

We shall now establish another couple of results for manipulating these distributivity 'pants'.

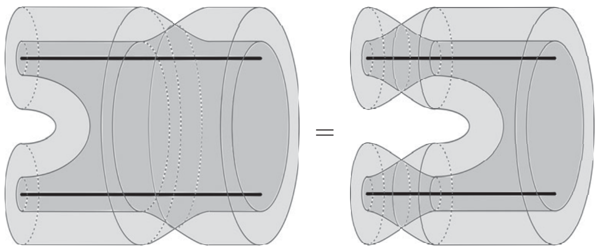
Lemma 5.1 (Mr. Bean's Pants Exchange†). The following diagram commutes:

$$\begin{tikzcd}
X \otimes (Y \otimes A) + X \otimes (Y \otimes B) \arrow[r, "\mathsf{dis}"] \arrow[d, "\alpha+\alpha"'] & X \otimes (Y \otimes A + Y \otimes B) \arrow[r, "X\otimes\mathsf{dis}"] & X \otimes (Y \otimes (A+B)) \arrow[d, "\alpha"] \\
(X \otimes Y) \otimes A + (X \otimes Y) \otimes B \arrow[d, "\gamma\otimes A+\gamma\otimes B"'] & & (X \otimes Y) \otimes (A+B) \arrow[d, "\gamma\otimes(A+B)"] \\
(Y \otimes X) \otimes A + (Y \otimes X) \otimes B \arrow[d, "\alpha^{-1}+\alpha^{-1}"'] & & (Y \otimes X) \otimes (A+B) \arrow[d, "\alpha^{-1}"] \\
Y \otimes (X \otimes A) + Y \otimes (X \otimes B) \arrow[r, "\mathsf{dis}"] & Y \otimes (X \otimes A + X \otimes B) \arrow[r, "Y\otimes\mathsf{dis}"] & Y \otimes (X \otimes (A+B))
\end{tikzcd}$$

and, in a strict monoidal category, is reduced to

$$\begin{tikzcd}
X \otimes Y \otimes A + X \otimes Y \otimes B \arrow[r, "\mathsf{dis}"] \arrow[d, "\gamma\otimes A+\gamma\otimes B"'] & X \otimes (Y \otimes A + Y \otimes B) \arrow[r, "X\otimes\mathsf{dis}"] & X \otimes Y \otimes (A+B) \arrow[d, "\gamma\otimes(A+B)"] \\
Y \otimes X \otimes A + Y \otimes X \otimes B \arrow[r, "\mathsf{dis}"] & Y \otimes (X \otimes A + X \otimes B) \arrow[r, "Y\otimes\mathsf{dis}"] & Y \otimes X \otimes (A+B)
\end{tikzcd}$$

Recall that γ denotes symmetry isomorphisms for $\otimes$, so in terms of tube diagrams, we have



(14)

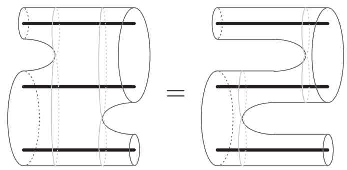
Proof. We draw two horizontal maps labelled with dis in the above diagram in the lemma, then use Lemma 2.2 (3) twice and naturality for dis from Lemma 2.1 (1). □

Next we observe that the interaction between distribution dis and the coproduct associativity α_+ from Lemma 2.2 (3) is equivalent to the following one in a strict monoidal setting:

$$\begin{tikzcd}
X \otimes A + X \otimes (B+C) \arrow[r, "\mathsf{dis}"] \arrow[d, "X\otimes A+\mathsf{dis}^{-1}"'] & X \otimes (A+B+C) \arrow[d, "\mathsf{dis}^{-1}"] \\
X \otimes A + X \otimes B + X \otimes C \arrow[r, "\mathsf{dis}+X\otimes C"'] & X \otimes (A+B) + X \otimes C
\end{tikzcd} \tag{15}$$

† *Mr. Bean*, Episode 1, Act 2, 1990.

In terms of tube diagrams, this is

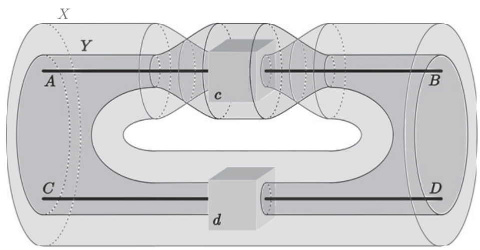


(16)

This equality of tubes can be found in McCurdy (2010).

Another operator we will make much use of in Section 6 is the additive parallel composition $\Box$. Following Definition 4.7, we compose the diagrams in (11) and (12) to obtain the following tube diagram for $c \,\Box\, d$:

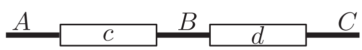
$c \,\Box\, d\;=$



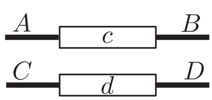
(17)

Remark 5.2. In order to turn our tube-diagram reasoning into mathematically rigorous proofs, we would need a coherence result of one form or another. It could be a statement that any non-strict such category is equivalent to a strict one; or a statement that the category of string/tube diagrams is the free such category. Currently, we have no such result, but this is not a total anomaly since among the dozens of well-known graphical languages for various kinds of (monoidal) categories collected in Selinger (2011), there are a number lacking coherence results, but they still offer useful guidelines for rigorous calculational proofs, much like the tube diagrams do in this paper.

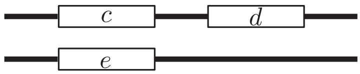
Remark 5.3. We should emphasise that all tube diagrams represent morphisms in the Kleisli category $\mathcal{K}\ell(T)$. We shall also employ a different kind of string diagram later, mostly for describing the trace axioms. This other kind of diagram is two-dimensional and, essentially, provides a 'pictorial shorthand' for the composition of components. For example, sequential composition $c \ggg d$ of components is represented by



additive parallel composition $c \,\Box\, d$ is represented by



and the 'identity component' $\mathsf{arr}(\mathrm{id})$ is represented simply by a wire/line. Hence the diagram



represents the composition $(c \,\Box\, e) \ggg (d \,\Box\, \mathsf{arr}(\mathrm{id}))$, which is equal to $(c \ggg d) \,\Box\, e$ up to a canonical isomorphism (this follows from the results in Asada and Hasuo (2010) and Hasuo *et al.* (2009)).

In this second type of string diagram, wires represent input/output interfaces and the state spaces of components are not explicit. There should be no problem distinguishing the two kinds of string diagrams. In particular, a component is represented in tube diagrams by a 3D shadowed cube, while in the second type of diagram it is a 2D box.

6. A monoidal trace for coalgebraic components

Jacobs (2010) showed how for certain monads T the Kleisli category $\mathscr{K}\ell(T)$ is traced monoidal with respect to coproducts $+$ as monoidal structure. Concretely, this means that for maps of the form $f : X + U \to T(Y + U)$, there is a trace map $\mathsf{Tr}^{\mathscr{K}\ell}(f) : X \to T(Y)$ satisfying standard properties (Joyal *et al.* 1996). This trace operator $\mathsf{Tr}^{\mathscr{K}\ell}$ on the Kleisli category will be used to construct a similar trace operator for coalgebraic components. The main task in this section is to show that the trace properties from Joyal *et al.* (1996) also hold for these components, but only up to (canonical) isomorphism.

The precise properties that T must satisfy to obtain this (Kleisli) trace operator $\mathsf{Tr}^{\mathscr{K}\ell}$ are listed in Jacobs (2010, Requirements 4.7). The main ones are that the category $\mathbb{C}$ should have (countable) coproducts, the Kleisli category should be enriched over the category of dcpo's with bottom and the monad should be 'semi-additive'. In this section we shall simply assume that these properties hold for T. Examples of such a monad T include the lift monad $T = 1 + (-)$ for partiality, the powerset monad $\mathscr{P}$ for non-determinism, as well as its bounded variant $\mathscr{P}^{<\kappa}$ with $\kappa > \aleph_0$, and the (discrete, countable) subdistribution monad $\mathscr{D}$ for probabilistic non-determinism where

$$\mathscr{D}X \;=\; \{d : X \to [0,1] \mid \textstyle\sum_{x\in X} d(x) \leqslant 1\}.$$

Such a d is a 'sub'distribution since its sum is $\leqslant 1$, rather than $= 1$ (see, for example, Hasuo *et al.* (2007)).

Definition 6.1. The trace operator

$$\mathsf{Tr} : (A + C, B + C) \to (A, B)$$

is defined on a coalgebraic component

$$c : X \otimes (A + C) \to T(X \otimes (B + C))$$

through the Kleisli trace operator $\mathsf{Tr}^{\mathscr{K}\ell}$

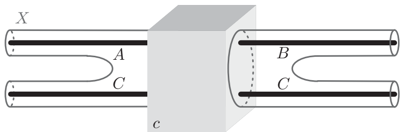
$$\mathsf{Tr}(c) \stackrel{\text{def}}{=} \mathsf{Tr}^{\mathscr{K}\ell}\Big(X \otimes A + X \otimes C \xrightarrow{T(\mathsf{dis}^{-1}) \circ c \circ \mathsf{dis}} T(X \otimes B + X \otimes C)\Big).$$

Note that this composition inside $\mathsf{Tr}^{\mathcal{K}\ell}(-)$ is really a Kleisli composition.

Using the tube diagram scheme described in Section 5, the trace operator can be depicted as follows. First, the composite inside the trace operator $\mathsf{Tr}^{\mathcal{K}\ell}$ is

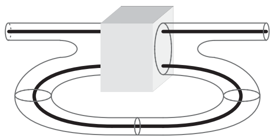
$$\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis} \quad : \quad X \otimes A + X \otimes B \longrightarrow X \otimes B + X \otimes C \qquad \text{in } \mathcal{K}\ell(T),$$

and is thus depicted by



Applying the trace operator $\mathsf{Tr}^{\mathcal{K}\ell}$ yields

$$\mathsf{Tr}(c) \; = \tag{18}$$



The functoriality of the operator Tr is essential here.

Lemma 6.2. The trace operator Tr extends to a functor

$$\mathsf{Tr} \quad : \quad \mathbf{Comp}(T, A + C, B + C) \longrightarrow \mathbf{Comp}(T, A, B).$$

That is, given two components

$$c : X \otimes (A + C) \to T(X \otimes (B + C))$$
$$d : Y \otimes (A + C) \to T(Y \otimes (B + C))$$

and a morphism f from c to d (see (4)), f is again a component morphism from

$$\mathsf{Tr}(c) : X \otimes A \to T(X \otimes B)$$

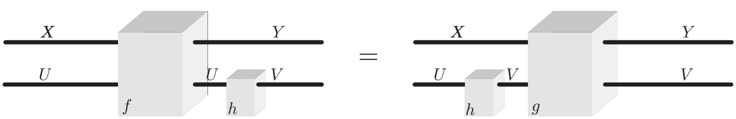
to

$$\mathsf{Tr}(d) : Y \otimes A \to T(Y \otimes B).$$

Proof. The proof makes essential use of the *uniformity* of the trace operator $\mathsf{Tr}^{\mathcal{K}\ell}$:

$$(\mathrm{id} + h) \bullet f = g \bullet (\mathrm{id} + h) \quad \text{implies} \quad \mathsf{Tr}^{\mathcal{K}\ell}(f) = \mathsf{Tr}^{\mathcal{K}\ell}(g), \tag{19}$$

Pictorially, this is



implies

Hasegawa first formulated this notion of uniformity for traced monoidal categories in Hasegawa (1999), and its name is derived by analogy with Plotkin's *uniformity principle* in domain theory (see, for example, Simpson and Plotkin (2000)) – for more recent developments as well as more on the historical background, see Hasegawa (2004).

It is typical that in a traced monoidal category $\mathbb{C}$, uniformity like (19) does not hold for every h but just for 'strict' h. However, when the trace structure of $\mathbb{C}$ arises from $\mathbb{C}$'s structure as a *partially additive category*, uniformity is true for every h (Haghverdi 2000). This is our current setting – see Jacobs (2010).

We now turn to the proof of the lemma. The following diagram in $\mathcal{K}\ell(T)$ commutes by the assumption that f is a component morphism, combined with the naturality of dis:

$$\begin{tikzcd}
X \otimes B + X \otimes C \arrow[r, "f\otimes B + f\otimes C"] & Y \otimes B + Y \otimes C \\
X \otimes (B + C) \arrow[u, "\mathsf{dis}^{-1}"] \arrow[r, "f\otimes(B+C)"] & Y \otimes (B + C) \arrow[u, "\mathsf{dis}^{-1}"'] \\
X \otimes (A + C) \arrow[u, "c"] \arrow[r, "f\otimes(A+C)"] & Y \otimes (A + C) \arrow[u, "d"'] \\
X \otimes A + X \otimes C \arrow[u, "\mathsf{dis}"] \arrow[r, "f\otimes A + f\otimes C"'] & Y \otimes A + Y \otimes C \arrow[u, "\mathsf{dis}"']
\end{tikzcd}$$

Thus we have

$$(\mathrm{id}+f\otimes C) \bullet \big((f\otimes B+\mathrm{id}) \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}\big) = \big(\mathsf{dis}^{-1} \bullet d \bullet \mathsf{dis} \bullet (f\otimes A+\mathrm{id})\big) \bullet (\mathrm{id}+f\otimes C),$$

from which we derive, by uniformity (19),

$$\mathsf{Tr}^{\mathcal{K}\ell}\big((f \otimes B + \mathrm{id}) \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}\big) = \mathsf{Tr}^{\mathcal{K}\ell}\big(\mathsf{dis}^{-1} \bullet d \bullet \mathsf{dis} \bullet (f \otimes A + \mathrm{id})\big). \tag{20}$$

This is used in the following calculation, which concludes the proof:

$$\begin{aligned}
(f \otimes B) \bullet \mathsf{Tr}(c) &= (f \otimes B) \bullet \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}) \\
&= \mathsf{Tr}^{\mathcal{K}\ell}\big((f \otimes B + \mathrm{id}) \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}\big) && \text{by the tightening axiom for} \\
& && \text{the trace operator } \mathsf{Tr}^{\mathcal{K}\ell} \\
& && \text{(see, for example, Joyal \textit{et al.} (1996))} \\
&= \mathsf{Tr}^{\mathcal{K}\ell}\big(\mathsf{dis}^{-1} \bullet d \bullet \mathsf{dis} \bullet (f \otimes A + \mathrm{id})\big) && \text{by (20)} \\
&= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet d \bullet \mathsf{dis}) \bullet (f \otimes A) && \text{by tightening} \\
&= \mathsf{Tr}(d) \bullet (f \otimes A). && \square
\end{aligned}$$

We will now make a special case explicit in the following lemma.

Lemma 6.3. For an isomorphism φ in $\mathbb{C}$ of the appropriate type,

$$(\varphi \otimes \mathrm{id}) \bullet \mathsf{Tr}(c) \bullet (\varphi^{-1} \otimes \mathrm{id}) \;=\; \mathsf{Tr}\big((\varphi \otimes \mathrm{id}) \bullet c \bullet (\varphi^{-1} \otimes \mathrm{id})\big).$$

Equivalently, if φ is an isomorphism of coalgebraic components as in the left-hand diagram below, then it is also an isomorphism in the right-hand diagram between the corresponding traces:

$$\begin{tikzcd} T(X \otimes (A+C)) \arrow[r, "T(\varphi\otimes\mathrm{id})", "\cong"'] & T(Y \otimes (A+C)) \\ X \otimes (A+C) \arrow[u, "c"] \arrow[r, "\varphi\otimes\mathrm{id}", "\cong"'] & Y \otimes (A+C) \arrow[u, "d"'] \end{tikzcd} \qquad \begin{tikzcd} T(X \otimes A) \arrow[r, "T(\varphi\otimes\mathrm{id})", "\cong"'] & T(Y \otimes A) \\ X \otimes A \arrow[u, "\mathsf{Tr}(c)"] \arrow[r, "\varphi\otimes\mathrm{id}", "\cong"'] & Y \otimes A \arrow[u, "\mathsf{Tr}(d)"'] \end{tikzcd}$$

□

In the remainder of this section, we first establish how the component trace Tr interacts with state extension $|$ and with additive parallel composition $\Box$. We will then prove the standard trace properties of Joyal *et al.* (1996). The trace properties, like the preceding lemmas, will often be accompanied by the corresponding equalities of tube diagrams, which we hope will convey some of the intuition behind the rather complicated calculations.

6.1. *Trace and state extension*

In Section 4.2 we have assumed that functors $X \otimes -$ preserve binary coproducts, for instance, because $X \otimes -$ has a right adjoint (given by exponents $\multimap$).

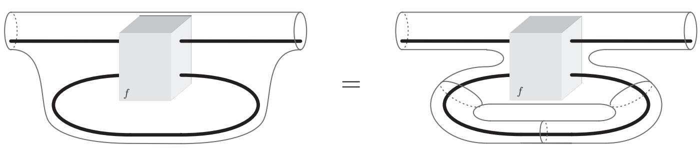
Proposition 6.4. Let T be a (commutative) monad on a symmetric monoidal category $\mathbb{C}$ with (countable) coproducts for which the Kleisli category $\mathcal{K}\ell(T)$ has a monoidal trace operator $\mathsf{Tr}^{\mathcal{K}\ell}$ with respect to coproducts. If functors $U \otimes - : \mathbb{C} \to \mathbb{C}$ preserve coproducts, then $U \otimes - : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$ preserves the trace operator in the sense that

$$U \otimes \mathsf{Tr}^{\mathcal{K}\ell}\Big(X + C \xrightarrow{\;f\;} T(Y+C)\Big) \;:\; U \otimes X \longrightarrow T(U \otimes Y)$$

is the same as

$$\mathsf{Tr}^{\mathcal{K}\ell}\Big(U \otimes X + U \otimes C \overset{\mathsf{dis}}{\underset{\cong}{\to}} U \otimes (X+C) \overset{U\otimes f}{\longrightarrow} T(U \otimes (Y+C)) \overset{T(\mathsf{dis}^{-1})}{\longrightarrow} T(U \otimes Y + U \otimes C)\Big).$$

In terms of tube diagrams,



Proof. The result follows from the way the monoidal trace is constructed through the coalgebraic trace in Jacobs (2010). First, the functor $Y + (-) : \mathbb{C} \to \mathbb{C}$ has an initial algebra in $\mathbb{C}$ given by the copower $\mathbb{N} \cdot Y$ with algebra map

$$\alpha_Y = [\kappa_0, [\kappa_{n+1}]_{n\in\mathbb{N}}] : Y + \mathbb{N} \cdot Y \overset{\cong}{\longrightarrow} \mathbb{N} \cdot Y.$$

The functor $U \otimes - : \mathbb{C} \to \mathbb{C}$ preserves coproducts by assumption, so the canonical map

$$d = [\mathrm{id} \otimes \kappa_n]_{n \in \mathbb{N}} : \mathbb{N} \cdot (U \otimes Y) \to U \otimes (\mathbb{N} \cdot Y)$$

is an isomorphism. It is then also an isomorphism of initial algebras.

The general trace theory in Hasuo *et al.* (2007) now says that $\mathbb{N} \cdot Y$ is the final coalgebra in the Kleisli category $\mathcal{K}\ell(T)$ of the functor $T(Y + (-))$. For a map

$$f : X + C \to T(Y + C)$$

we first take

$$\widehat{f} = T(\mathrm{id} + \kappa_2) \circ f : X + C \to Y + (X + C),$$

which yields a unique map

$$\mathsf{beh}(\widehat{f}) : X + C \to T(\mathbb{N} \cdot Y)$$

to the final coalgebra, and finally the trace map itself as

$$\mathsf{Tr}^{\mathcal{K}\ell}(f) = T(\nabla) \circ \mathsf{beh}(\widehat{f}) \circ \kappa_1 : X \to T(Y).$$

We can similarly obtain a trace map

$$\mathsf{Tr}^{\mathcal{K}\ell}(f_U) : U \otimes X \to U \otimes Y$$

for the morphism

$$f_U = T(\mathsf{dis}^{-1}) \circ (U \otimes f) \circ \mathsf{dis} : U \otimes X + U \otimes C \to T(U \otimes Y + U \otimes X)$$

used in the proposition. We are then done if the following diagram commutes:

$$\begin{tikzcd}
& U \otimes X + U \otimes C \arrow[r, "\mathsf{beh}(\widehat{f_U})"'] \arrow[d, "\mathsf{dis}"', "\cong"] & T(\mathbb{N} \cdot (U \otimes Y)) \arrow[d, "\cong"', "T(d)"] \arrow[dr, "T(\nabla)"] & \\
U \otimes X \arrow[ur, "\kappa_1"] \arrow[dr, "U \otimes \kappa_1"'] \arrow[rrr, bend left=60, "\mathsf{Tr}^{\mathcal{K}\ell}(f_U)"] \arrow[rrr, bend right=60, "U \otimes \mathsf{Tr}^{\mathcal{K}\ell}(f)"'] & & & T(U \otimes Y) \\
& U \otimes (X + C) \arrow[r, "U \otimes \mathsf{beh}(\widehat{f})"] & T(U \otimes \mathbb{N} \cdot Y) \arrow[ur, "T(\mathrm{id} \otimes \nabla)"'] &
\end{tikzcd}$$

It is obvious that the two triangles commute, and commutation of the inner rectangle follows by a finality argument, in the Kleisli category:

$$\begin{tikzcd}
U \otimes Y + (U \otimes X + U \otimes Y) \arrow[r, Rightarrow] & U \otimes Y + U \otimes \mathbb{N} \cdot Y \\
U \otimes Y + U \otimes C \arrow[u, "\mathrm{id}+\kappa_2"] & U \otimes (Y + \mathbb{N} \cdot Y) \arrow[u, "\cong"', "\mathsf{dis}^{-1}"] \\
U \otimes X + U \otimes C \arrow[u, "f_U"] \arrow[r, Rightarrow, shift left, "(U \otimes \mathsf{beh}(\widehat{f})) \circ \mathsf{dis}"] \arrow[r, Rightarrow, shift right, "T(d) \circ \mathsf{beh}(\widehat{f_U})"'] & U \otimes \mathbb{N} \cdot Y \arrow[u, "\cong"', "U \otimes \alpha_Y^{-1}"]
\end{tikzcd}$$

The remaining details are left as an exercise. □

We now return to our framework of coalgebraic components, and show how trace and state extension interact.

Lemma 6.5. Trace commutes with state extension:

$$U \mid \mathsf{Tr}(c) = \mathsf{Tr}(U \mid c)$$
$$\mathsf{Tr}(c) \mid V = \mathsf{Tr}(c \mid V).$$

Proof. We use several standard properties of the trace $\mathsf{Tr}^{\mathcal{K}\ell}$ in the Kleisli category $\mathcal{K}\ell(T)$, such as (di)naturality, but we mainly depend on Proposition 6.4. We calculate in this Kleisli category:

$$\begin{aligned}
U \mid \mathsf{Tr}(c) &= \alpha \bullet (U \otimes \mathsf{Tr}(c)) \bullet \alpha^{-1} \\
&= \alpha \bullet (U \otimes \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis})) \bullet \mathsf{dis})) \bullet \alpha^{-1} \\
&= \alpha \bullet \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet U \otimes (\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}) \bullet \mathsf{dis}) \bullet \alpha^{-1} \qquad \text{by Proposition 6.4} \\
&= \mathsf{Tr}^{\mathcal{K}\ell}((\alpha + \mathrm{id}) \bullet \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet (U \otimes c) \bullet \\
&\qquad (U \otimes \mathsf{dis}) \bullet \mathsf{dis} \bullet (\alpha^{-1} + \mathrm{id})) \qquad \text{by the naturality of dis} \\
&\qquad\qquad \text{(see Lemma 2.1 (1))} \\
&= \mathsf{Tr}^{\mathcal{K}\ell}((\mathrm{id} + \alpha^{-1}) \bullet (\mathrm{id} + \alpha) \bullet (\alpha + \mathrm{id}) \bullet \\
&\qquad \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet (U \otimes c) \bullet (U \otimes \mathsf{dis}) \bullet \mathsf{dis} \bullet (\alpha^{-1} + \mathrm{id})) \\
&= \mathsf{Tr}^{\mathcal{K}\ell}((\alpha + \alpha) \bullet \mathsf{dis}^{-1} \bullet (U \otimes \mathsf{dis}^{-1}) \bullet (U \otimes c) \bullet \\
&\qquad (U \otimes \mathsf{dis}) \bullet \mathsf{dis} \bullet (\alpha^{-1} + \alpha^{-1})) \qquad \text{by the dinaturality of } \mathsf{Tr}^{\mathcal{K}\ell} \\
&= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet \alpha \bullet (U \otimes c) \bullet \alpha^{-1} \bullet \mathsf{dis}) \qquad \text{by Lemma 2.1 (2)} \\
&= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet (U \mid c) \bullet \mathsf{dis}) \\
&= \mathsf{Tr}(U \mid c).
\end{aligned}$$

We immediately use this property in

$$\begin{aligned}
\mathsf{Tr}(c) \mid V &= (\gamma \otimes \mathrm{id}) \bullet (V \mid \mathsf{Tr}(c)) \bullet (\gamma \otimes \mathrm{id}) \\
&= (\gamma \otimes \mathrm{id}) \bullet \mathsf{Tr}(V \mid c) \bullet (\gamma \otimes \mathrm{id}) \qquad \text{as just proved} \\
&= \mathsf{Tr}((\gamma \otimes \mathrm{id}) \bullet (V \mid c) \bullet (\gamma \otimes \mathrm{id})) \qquad \text{by Lemma 6.3} \\
&= \mathsf{Tr}(c \mid V).
\end{aligned}$$

□

6.2. *Trace and additive parallel composition*

The following lemma describes the interaction of trace and additive parallel composition. It will be crucial for proving the (di)naturality properties for Tr in Section 6.3, which involve composition $\ggg$ of components instead of Kleisli composition $\bullet$.

Lemma 6.6. For appropriately typed components c, d, we have:

(1) $\mathsf{Tr}((d \,\Box\, \mathsf{arr}(\mathrm{id})) \bullet c) = (d \mid I) \bullet \mathsf{Tr}(c)$.

(2) $\mathsf{Tr}(c \bullet (d \,\Box\, \mathsf{arr}(\mathrm{id}))) = \mathsf{Tr}(c) \bullet (d \mid I)$.
(3) $\mathsf{Tr}((\mathsf{arr}(\mathrm{id}) \,\Box\, d) \bullet c) = \mathsf{Tr}(c \bullet (\mathsf{arr}(\mathrm{id}) \,\Box\, d))$.

Proof. We shall prove (1) and (3); the proof of (2) is similar to the proof of (1).

— For (1):

$$\begin{aligned}
\mathsf{Tr}((d \,\Box\, \mathsf{arr}(\mathrm{id})) \bullet c) &= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet \mathsf{dis} \bullet (d \mid I + Y \mid \mathsf{arr}(\mathrm{id})) \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}) \\
&= \mathsf{Tr}^{\mathcal{K}\ell}((d \mid I + \mathrm{id}) \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}) \\
&= (d \mid I) \bullet \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}) \qquad \text{by naturality of } \mathsf{Tr}^{\mathcal{K}\ell} \\
&= (d \mid I) \bullet \mathsf{Tr}(c).
\end{aligned}$$

— For (3):

$$\begin{aligned}
\mathsf{Tr}((\mathsf{arr}(\mathrm{id}) \,\Box\, d) \bullet c) &= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet \mathsf{dis} \bullet (\mathsf{arr}(\mathrm{id}) \mid Y + I \mid d) \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}^{-1}) \\
&= \mathsf{Tr}^{\mathcal{K}\ell}((\mathrm{id} + I \mid d) \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}^{-1}) \\
&= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}^{-1} \bullet (\mathrm{id} + I \mid d)) \quad \text{by dinaturality of } \mathsf{Tr}^{\mathcal{K}\ell} \\
&= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}^{-1} \bullet (\mathsf{arr}(\mathrm{id}) \mid Y + I \mid d) \bullet \mathsf{dis}^{-1} \bullet \mathsf{dis}) \\
&= \mathsf{Tr}(c \bullet (\mathsf{arr}(\mathrm{id}) \,\Box\, d)).
\end{aligned}$$
$\square$

6.3. *Trace axioms*

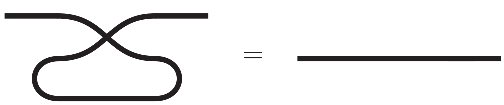
In this final part of Section 6 we verify the trace axioms from Joyal *et al.* (1996), formulated in a component setting (with explicit isomorphisms). For each axiom, drawing a pictorial (pseudo-)proof with tube diagrams was helpful in developing a rigorous, calculational proof. We will just present such a pictorial proof for one property, namely the Post-Composition Naturality property, as an example.

Yanking

In the language of components, the Yanking property can be formulated as a diagram of the form

$$\begin{tikzcd}
1 \arrow[rr, "\mathsf{arr}(\gamma_+)"] \arrow[drr, "\mathsf{arr}(\mathrm{id})"'] & & (A+A, A+A) \arrow[d, "\mathsf{Tr}"] \\
& & (A, A)
\end{tikzcd}$$

where $\gamma_+ : A + A \xrightarrow{\cong} A + A$ is the monoidal swap map associated with coproducts $+$ (see Lemma 2.2 (2)). Pictorially, we have



We need to show that the Kleisli trace of the map

$$I \otimes A + I \otimes A \xrightarrow[\cong]{\mathsf{dis}} I \otimes (A + A) \xrightarrow{\mathsf{arr}(\gamma_+)} T(I \otimes (A + A)) \xrightarrow[\cong]{T(\mathsf{dis}^{-1})} T(I \otimes A + I \otimes A)$$

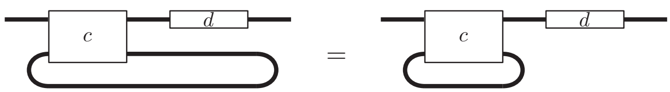
is the (Kleisli) identity $\eta = \mathsf{arr}(\mathrm{id})$. This, together with the yanking property for the Kleisli trace operator $\mathsf{Tr}^{\mathscr{K}\ell}$, will be used in

$$\begin{aligned}
\mathsf{Tr}(\mathsf{arr}(\gamma)) &= \mathsf{Tr}^{\mathscr{K}\ell}\big(T(\mathsf{dis}^{-1}) \circ \mathsf{arr}(\gamma_+) \circ \mathsf{dis}\big) \\
&= \mathsf{Tr}^{\mathscr{K}\ell}\big(T(\mathsf{dis}^{-1}) \circ \eta \circ (\mathrm{id} \otimes \gamma_+) \circ \mathsf{dis}]\big) \\
&\overset{2.2(2)}{=} \mathsf{Tr}^{\mathscr{K}\ell}\big(\eta \circ \mathsf{dis}^{-1} \circ \mathsf{dis} \circ \gamma_+\big) \\
&= \mathsf{Tr}^{\mathscr{K}\ell}\big(\eta \circ \gamma_+\big) = \mathsf{Tr}^{\mathscr{K}\ell}\big(\gamma_+^{\mathscr{K}\ell}\big) = \mathrm{id}^{\mathscr{K}\ell}_{I \otimes A} = \eta_{I \otimes A} = \mathsf{arr}(\mathrm{id}_A).
\end{aligned}$$

Post-composition naturality/tightening

$$\begin{tikzcd}
(A + C, B + C) \times (B, D) \arrow[r, "\mathsf{Tr} \times \mathrm{id}"] \arrow[d] & (A, B) \times (B, D) \arrow[r, "\ggg"] & (A, D) \\
(A + C, B + C) \times ((B, D) \times (C, C)) \arrow[d, "\mathrm{id} \times \Box"'] & & \\
(A + C, B + C) \times (B + C, D + C) \arrow[rr, "\ggg"] & & (A + C, D + C) \arrow[uu, "\mathsf{Tr}"']
\end{tikzcd}$$

The usual string representation of this axiom is



(21)

The aim is to prove for

$$X \otimes (A + C) \xrightarrow{c} T(X \otimes (B + C))$$

$$Y \otimes B \xrightarrow{d} T(Y \otimes D)$$

that the following diagram commutes:

$$\begin{tikzcd}
T((X \otimes (Y \otimes I)) \otimes D) \arrow[r, "T((\mathrm{id} \otimes \rho) \otimes \mathrm{id})", "\cong"'] & T((X \otimes Y) \otimes D) \\
(X \otimes (Y \otimes I)) \otimes A \arrow[u, "\mathsf{Tr}(c \ggg (d \Box \mathsf{arr}(\mathrm{id})))"] \arrow[r, "\cong", "(\mathrm{id} \otimes \rho) \otimes \mathrm{id}"'] & (X \otimes Y) \otimes A \arrow[u, "\mathsf{Tr}(c) \ggg d"']
\end{tikzcd}$$

We shall make crucial use of Lemma 6.6, but the rest is mainly bookkeeping.

$$
\begin{aligned}
& T((\mathrm{id}\otimes\rho)\otimes\mathrm{id})\bullet\mathsf{Tr}(c \ggg (d\,\Box\,\mathsf{arr}(\mathrm{id}))) \\
&= (\rho\otimes\mathrm{id})\bullet(\alpha\otimes\mathrm{id})\bullet\mathsf{Tr}((X\mid(d\,\Box\,\mathsf{arr}(\mathrm{id})))\bullet(c\mid(Y\otimes I))) \\
&\stackrel{6.3}{=} (\rho\otimes\mathrm{id})\bullet\mathsf{Tr}\Big((\alpha\otimes\mathrm{id})\bullet(X\mid(d\,\Box\,\mathsf{arr}(\mathrm{id})))\bullet(c\mid(Y\otimes I))\bullet(\alpha^{-1}\otimes\mathrm{id})\Big) \\
&\qquad\quad \bullet(\alpha\otimes\mathrm{id}) \\
&\stackrel{4.8}{=} (\rho\otimes\mathrm{id})\bullet\mathsf{Tr}\Big(((X\mid d)\,\Box\,\mathsf{arr}(\mathrm{id}))\bullet(\alpha\otimes\mathrm{id})\bullet(c\mid(Y\otimes I))\bullet(\alpha^{-1}\otimes\mathrm{id})\Big) \\
&\qquad\quad \bullet(\alpha\otimes\mathrm{id}) \\
&\stackrel{3.2\,(5)}{=} (\rho\otimes\mathrm{id})\bullet\mathsf{Tr}\Big(((X\mid d)\,\Box\,\mathsf{arr}(\mathrm{id}))\bullet((c\mid Y)\mid I)\Big)\bullet(\alpha\otimes\mathrm{id}) \\
&\stackrel{6.6}{=} (\rho\otimes\mathrm{id})\bullet((X\mid d)\mid I)\bullet\mathsf{Tr}\Big((c\mid Y)\mid I\Big)\bullet(\alpha\otimes\mathrm{id}) \\
&\stackrel{6.5}{=} (\rho\otimes\mathrm{id})\bullet((X\mid d)\mid I)\bullet((\mathsf{Tr}(c)\mid Y)\mid I)\bullet(\alpha\otimes\mathrm{id}) \\
&\stackrel{3.2\,(3)}{=} (\rho\otimes\mathrm{id})\bullet\Big(((X\mid d)\bullet(\mathsf{Tr}(c)\mid Y))\mid I\Big)\bullet(\alpha\otimes\mathrm{id}) \\
&\stackrel{3.2\,(4)}{=} ((X\mid d)\bullet(\mathsf{Tr}(c)\mid Y))\bullet(\rho\otimes\mathrm{id})\bullet(\alpha\otimes\mathrm{id}) \\
&= (\mathsf{Tr}(c)\ggg d)\circ((\mathrm{id}\otimes\rho)\otimes\mathrm{id}).
\end{aligned}
$$

A pictorial (pseudo-)proof of the property is presented in Figure 1. Although Lemmas 6.3, 6.5 and 6.6 are useful in the above calculational proof, more basic properties such as Lemma 5.1, on which Lemmas 6.3, 6.5 and 6.6 rely, have clearer pictorial meanings. Therefore the latter are used in the pictorial proof.

Pre-composition naturality/tightening

$$
\begin{tikzcd}
(D,A)\times(A+C,B+C) \arrow[r, "\mathrm{id}\times\mathsf{Tr}"] \arrow[d] & (D,A)\times(A,B) \arrow[r, "\ggg"] & (D,B) \\
((D,A)\times(C,C))\times(A+C,B+C) \arrow[d, "\Box\times\mathrm{id}"'] & & \\
(D+C,A+C)\times(A+C,B+C) \arrow[rr, "\ggg"] & & (D+C,B+C) \arrow[uu, "\mathsf{Tr}"']
\end{tikzcd}
$$

The aim is to prove for

$$X\otimes(A+C)\xrightarrow{c}T(X\otimes(B+C))$$

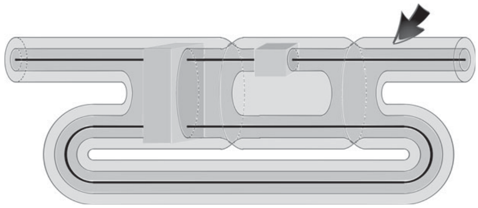
$$Y\otimes D\xrightarrow{d}T(Y\otimes A)$$

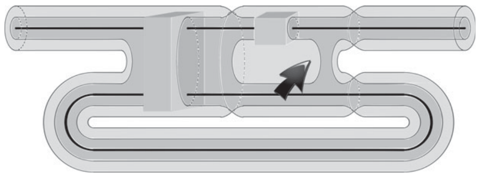
that the following diagram commutes:

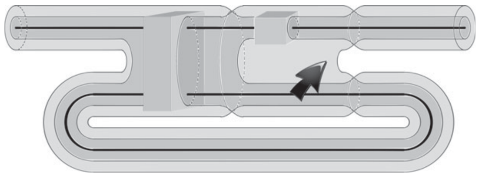
$$
\begin{tikzcd}
T(((Y\otimes I)\otimes X)\otimes B) \arrow[r, "T((\rho\otimes\mathrm{id})\otimes\mathrm{id})", "\cong"'] & T((Y\otimes X)\otimes B) \\
((Y\otimes I)\otimes X))\otimes A \arrow[u, "\mathsf{Tr}((d\Box\mathsf{arr}(\mathrm{id}))\ggg c)"] \arrow[r, "\cong", "(\rho\otimes\mathrm{id})\otimes\mathrm{id}"'] & (Y\otimes X)\otimes A \arrow[u, "d\ggg\mathsf{Tr}(c)"']
\end{tikzcd}
$$

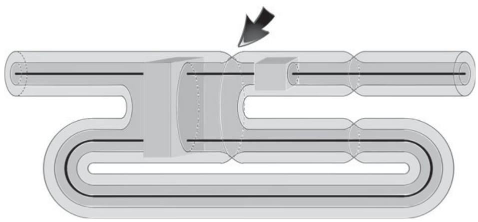
This is left as an exercise.

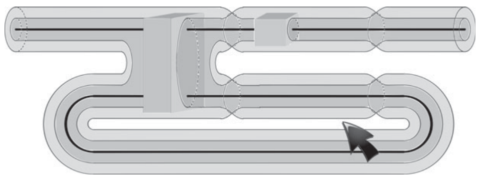
$\mathsf{Tr}\big(c \ggg (d \,\square\, \mathsf{arr}(\mathrm{id}))\big)$

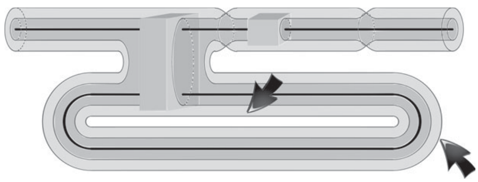
$=$ 

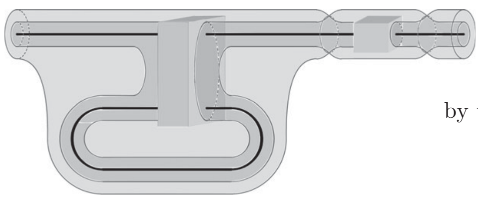
$=$  by Lemma 5.1

$=$  by (13)

$=$  by the naturality of dis

$=$  by Lemma 5.1

$=$  'waist' symmetries γ are isomorphisms

$=$  by the post-composition naturality of $\mathsf{Tr}^{\mathcal{K}\ell}$, and Proposition 6.4

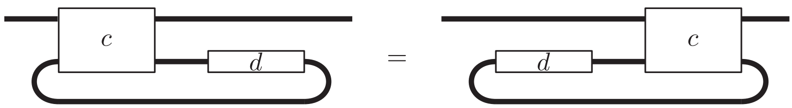
$= \quad \mathsf{Tr}(c) \ggg d$

Fig. 1. A pictorial proof of post-composition naturality. The pointers show where the transformation is going to occur.

Dinaturality

$$\begin{tikzcd}
& (A+C,B+D)\times(B+D,B+C) \arrow[dr, "\ggg"] & \\
(A+C,B+D)\times((B,B)\times(D,C)) \arrow[ur, "\mathrm{id}\times\Box"'] & & (A+C,B+C) \arrow[d, "\mathsf{Tr}"] \\
(A+C,B+D)\times(D,C) \arrow[u] \arrow[d, equal, "\wr"] & & (A,B) \arrow[d, equal] \\
(D,C)\times(A+C,B+D) \arrow[d] & & (A,B) \\
((A,A)\times(D,C))\times(A+C,B+D) \arrow[dr, "\Box\times\mathrm{id}"] & & (A+D,B+D) \arrow[u, "\mathsf{Tr}"'] \\
& (A+D,A+C)\times(A+C,B+D) \arrow[ur, "\ggg"] &
\end{tikzcd}$$

In terms of string diagrams, we have



For coalgebras

$$X\otimes(A+C)\xrightarrow{c}T(X\otimes(B+D))$$

$$Y\otimes D\xrightarrow{d}T(Y\otimes C)$$

we need to show that the following diagram commutes:

$$\begin{tikzcd}
T((X\otimes(I\otimes Y))\otimes B) \arrow[r, "T(\gamma\otimes\mathrm{id})", "\cong"'] & T(((I\otimes Y)\otimes X)\otimes B) \\
(X\otimes(I\otimes Y))\otimes A \arrow[u, "\mathsf{Tr}(c\ggg(\mathsf{arr}(\mathrm{id}_B)\Box d))"] \arrow[r, "\cong", "T(\gamma\otimes\mathrm{id})"'] & ((I\otimes Y)\otimes X)\otimes A \arrow[u, "\mathsf{Tr}((\mathsf{arr}(\mathrm{id}_A)\Box d)\ggg c)"']
\end{tikzcd}$$

The essence again lies in Lemma 6.6, but with quite a lot of bookkeeping this time:

$$
\begin{aligned}
&(\gamma \otimes \mathrm{id}) \bullet \mathsf{Tr}\big(c \ggg (\mathsf{arr}(\mathrm{id}_B) \,\square\, d)\big) \\
&\overset{6.3}{=} \mathsf{Tr}\Big((\gamma \otimes \mathrm{id}) \bullet (X \mid (\mathsf{arr}(\mathrm{id}_B) \,\square\, d)) \bullet ((I \otimes Y) \mid c) \bullet (\gamma \otimes \mathrm{id})\Big) \bullet (\gamma \otimes \mathrm{id}) \\
&= (\alpha \otimes \mathrm{id}) \bullet (\alpha^{-1} \otimes \mathrm{id}) \bullet \mathsf{Tr}\Big(((\mathsf{arr}(\mathrm{id}_B) \,\square\, d) \mid X) \bullet (c \mid (I \otimes Y))\Big) \bullet (\gamma \otimes \mathrm{id}) \\
&\overset{6.3}{=} (\alpha \otimes \mathrm{id}) \bullet \mathsf{Tr}\Big((\alpha^{-1} \otimes \mathrm{id}) \bullet ((\mathsf{arr}(\mathrm{id}_B) \,\square\, d) \mid X) \bullet (c \mid (I \otimes Y)) \bullet (\alpha \otimes \mathrm{id})\Big) \bullet \\
&\qquad (\alpha^{-1} \otimes \mathrm{id}) \bullet (\gamma \otimes \mathrm{id}) \\
&\overset{4.8}{=} (\alpha \otimes \mathrm{id}) \bullet \mathsf{Tr}\Big((\mathsf{arr}(\mathrm{id}_B) \,\square\, (d \mid X)) \bullet (\alpha^{-1} \otimes \mathrm{id}) \bullet (c \mid (I \otimes Y)) \bullet (\alpha \otimes \mathrm{id})\Big) \bullet \\
&\qquad (\alpha^{-1} \otimes \mathrm{id}) \bullet (\gamma \otimes \mathrm{id}) \\
&\overset{6.6}{=} (\alpha \otimes \mathrm{id}) \bullet \mathsf{Tr}\Big((\alpha^{-1} \otimes \mathrm{id}) \bullet (c \mid (I \otimes Y)) \bullet (\alpha \otimes \mathrm{id}) \bullet (\mathsf{arr}(\mathrm{id}_B) \,\square\, (d \mid X))\Big) \bullet \\
&\qquad (\alpha^{-1} \otimes \mathrm{id}) \bullet (\gamma \otimes \mathrm{id}) \\
&\overset{4.8}{=} (\alpha \otimes \mathrm{id}) \bullet \mathsf{Tr}\Big((\alpha^{-1} \otimes \mathrm{id}) \bullet (c \mid (I \otimes Y)) \bullet ((\mathsf{arr}(\mathrm{id}_B) \,\square\, d) \mid X)) \bullet (\alpha \otimes \mathrm{id})\Big) \bullet \\
&\qquad (\alpha^{-1} \otimes \mathrm{id}) \bullet (\gamma \otimes \mathrm{id}) \\
&\overset{6.3}{=} (\alpha \otimes \mathrm{id}) \bullet (\alpha^{-1} \otimes \mathrm{id}) \bullet \mathsf{Tr}\Big((c \mid (I \otimes Y)) \bullet ((\mathsf{arr}(\mathrm{id}_B) \,\square\, d) \mid X))\Big) \bullet \\
&\qquad (\gamma \otimes \mathrm{id}) \\
&= \mathsf{Tr}\big((\mathsf{arr}(\mathrm{id}_B) \,\square\, d) \ggg c\big) \bullet (\gamma \otimes \mathrm{id}).
\end{aligned}
$$

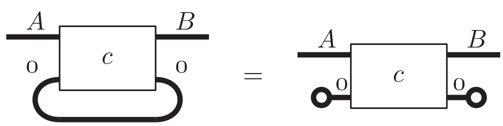
Unit vanishing

The relevant component diagram is

$$
\begin{tikzcd}
(A+0, B+0) \arrow[rr, "\mathsf{Tr}"] \arrow[d, "\langle \mathsf{arr}(\rho_+^{-1}), \mathrm{id}, \mathsf{arr}(\rho_+) \rangle"'] & & (A, B) \\
(A, A+0) \times (A+0, B+0) \times (B+0, B) \arrow[rr, "\ggg \times \mathrm{id}"'] & & (A, B+0) \times (B+0, B) \arrow[u, "\ggg"']
\end{tikzcd}
$$

where we have to bear in mind that the $\rho_+ : C + 0 \xrightarrow{\cong} C$ refers to the monoidal isomorphism with respect to the coproducts $+$ on $\mathbb{C}$.

Pictorially, the axiom asserts



We will prove that there is an isomorphism of components:

$$
\begin{tikzcd}
T(((I \otimes X) \otimes I) \otimes B) \arrow[rr, "T((\lambda \circ \rho) \otimes \mathrm{id})", "\cong"'] & & T(X \otimes B) \\
((I \otimes X) \otimes I) \otimes A \arrow[rr, "\cong", "(\lambda \circ \rho) \otimes \mathrm{id}"'] \arrow[u, "(\mathsf{arr}(\rho_+^{-1}) \ggg c) \ggg \mathsf{arr}(\rho_+)"] & & X \otimes A \arrow[u, "\mathsf{Tr}(c)"']
\end{tikzcd}
$$

The heart of the matter is:

$$\begin{aligned}
\mathsf{Tr}(c) &= \mathsf{Tr}^{\mathcal{K}\ell}(\mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis}) && \text{with } \mathsf{dis} : (X \otimes A) + (X \otimes 0) \xrightarrow{\cong} X \otimes (A + 0) \\
&= \rho_+ \bullet \mathsf{dis}^{-1} \bullet c \bullet \mathsf{dis} \bullet \rho_+^{-1} && \text{by vanishing for } \mathsf{Tr}^{\mathcal{K}\ell}\text{, since } X \otimes 0 \text{ is initial} \\
&= (\mathrm{id} \otimes \rho_+^{-1}) \bullet c \bullet (\mathrm{id} \otimes \rho_+) && \text{by Lemma 2.2 (1).}
\end{aligned}$$

Hence we obtain the required isomorphism of components:

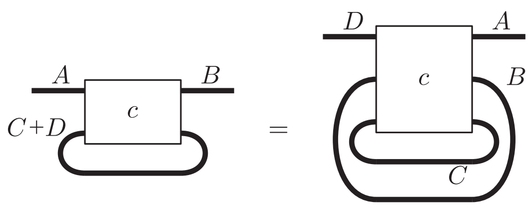
$$\begin{aligned}
&((\lambda \circ \rho) \otimes \mathrm{id}) \bullet \big((\mathsf{arr}(\rho_+^{-1}) \ggg c) \ggg \mathsf{arr}(\rho_+)\big) \\
&\quad \overset{4.2\,(2)}{=} (\lambda \otimes \mathrm{id}) \bullet (\mathrm{id} \otimes \rho_+) \bullet (\mathsf{arr}(\rho_+^{-1}) \ggg c) \bullet (\rho \otimes \mathrm{id}) \\
&\quad = (\mathrm{id} \otimes \rho_+) \bullet (\lambda \otimes \mathrm{id}) \bullet (\mathsf{arr}(\rho_+^{-1}) \ggg c) \bullet (\rho \otimes \mathrm{id}) \\
&\quad \overset{4.2\,(2)}{=} (\mathrm{id} \otimes \rho_+) \bullet c \bullet (\mathrm{id} \otimes \rho_+^{-1}) \bullet (\lambda \otimes \mathrm{id}) \bullet (\rho \otimes \mathrm{id}) \\
&\quad = \mathsf{Tr}(c) \bullet ((\lambda \circ \rho) \otimes \mathrm{id}), \qquad \text{as shown above.}
\end{aligned}$$

Tensor vanishing

Again we have to distinguish carefully between the monoidal associativity isomorphisms α_+ and α for coproduct $+$ and tensor $\otimes$, respectively. The component diagram is

$$\begin{tikzcd}
((A+C)+D, (B+C)+D) \arrow[r, "\langle \mathsf{arr}(\alpha_+), \mathrm{id}, \mathsf{arr}(\alpha_+^{-1}) \rangle"] \arrow[d, "\mathsf{Tr}"'] & \begin{array}{l}(A+(C+D), (A+C)+D) \times \\ ((A+C)+D, (B+C)+D) \times \\ ((B+C)+D, B+(C+D))\end{array} \arrow[dd, "(\ggg \times \mathrm{id}) \circ \ggg"] \\
(A+C, B+C) \arrow[d, "\mathsf{Tr}"'] & \\
(A, B) & (A+(C+D), B+(C+D)) \arrow[l, "\mathsf{Tr}"']
\end{tikzcd}$$

Pictorially, we have



Our aim is to prove, for a component

$$X \otimes ((A+C)+D) \xrightarrow{c} X \otimes ((B+C)+D),$$

that we have an isomorphism of components:

$$\begin{tikzcd}
T(((I \otimes X) \otimes I) \otimes A) \arrow[r, "T((\lambda \circ \rho) \otimes \mathrm{id})", "\cong"'] & T(X \otimes A) \\
((I \otimes X) \otimes I) \otimes A \arrow[u, "\mathsf{Tr}((\mathsf{arr}(\alpha_+) \ggg c) \ggg \mathsf{arr}(\alpha_+^{-1}))"] \arrow[r, "\cong", "(\lambda \circ \rho) \otimes \mathrm{id}"'] & X \otimes A \arrow[u, "\mathsf{Tr}(\mathsf{Tr}(c))"']
\end{tikzcd}$$

This is done as follows.

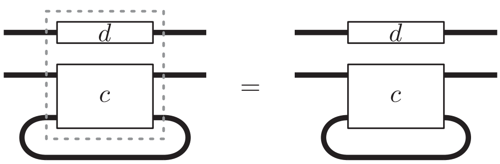
$$
\begin{aligned}
&((\lambda \circ \rho) \otimes \mathrm{id}) \bullet \mathsf{Tr}((\mathsf{arr}(\alpha_+) \ggg c) \ggg \mathsf{arr}(\alpha_+^{-1}))\\
&\quad \overset{6.3}{=} \mathsf{Tr}\big(((\lambda \circ \rho) \otimes \mathrm{id}) \bullet (\mathsf{arr}(\alpha_+) \ggg c) \ggg \mathsf{arr}(\alpha_+^{-1}) \bullet ((\rho^{-1} \circ \lambda^{-1}) \otimes \mathrm{id})\big)\\
&\qquad\qquad \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\quad \overset{4.2\,(2)}{=} \mathsf{Tr}\big((\lambda \otimes \mathrm{id}) \bullet (\mathrm{id} \otimes \alpha_+^{-1}) \bullet (\mathsf{arr}(\alpha_+) \ggg c) \bullet (\lambda^{-1} \otimes \mathrm{id})\big) \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\quad = \mathsf{Tr}\big((\mathrm{id} \otimes \alpha_+^{-1}) \bullet (\lambda \otimes \mathrm{id}) \bullet (\mathsf{arr}(\alpha_+) \ggg c) \bullet (\lambda^{-1} \otimes \mathrm{id})\big) \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\quad \overset{4.2\,(2)}{=} \mathsf{Tr}\big((\mathrm{id} \otimes \alpha_+^{-1}) \bullet c \bullet (\mathrm{id} \otimes \alpha_+)\big) \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\quad = \mathsf{Tr}^{\mathcal{K}\ell}\big(\,\mathsf{dis}^{-1} \bullet (\mathrm{id} \otimes \alpha_+^{-1}) \bullet c \bullet (\mathrm{id} \otimes \alpha_+) \bullet \mathsf{dis}\,\big) \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\quad \overset{2.2\,(3)}{=} \mathsf{Tr}^{\mathcal{K}\ell}\big(\,\mathsf{dis}^{-1} \bullet (\mathrm{id} \otimes \alpha_+^{-1}) \bullet c \bullet \mathsf{dis} \bullet (\mathsf{dis} + \mathrm{id}) \bullet \alpha_+ \bullet (\mathrm{id} + \mathsf{dis}^{-1})\big)\\
&\qquad\qquad \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\quad = \mathsf{Tr}^{\mathcal{K}\ell}\big((\mathrm{id} + \mathsf{dis}^{-1}) \bullet \mathsf{dis}^{-1} \bullet (\mathrm{id} \otimes \alpha_+^{-1}) \bullet c \bullet \mathsf{dis} \bullet (\mathsf{dis} + \mathrm{id}) \bullet \alpha_+\big)\\
&\qquad\qquad \bullet ((\lambda \circ \rho) \otimes \mathrm{id}) \qquad\qquad \text{by dinaturality for } \mathsf{Tr}^{\mathcal{K}\ell}\\
&\quad \overset{2.2\,(3)}{=} \mathsf{Tr}^{\mathcal{K}\ell}\big(\alpha_+^{-1} \bullet (\mathsf{dis}^{-1} + \mathrm{id}) \bullet \mathsf{dis} \bullet c \bullet \mathsf{dis} \bullet (\mathsf{dis} + \mathrm{id}) \bullet \alpha_+\big) \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\quad = \mathsf{Tr}^{\mathcal{K}\ell}\big(\mathsf{Tr}^{\mathcal{K}\ell}\big((\mathsf{dis}^{-1} + \mathrm{id}) \bullet \mathsf{dis} \bullet c \bullet \mathsf{dis} \bullet (\mathsf{dis} + \mathrm{id})\big)\big) \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by vanishing for } \mathsf{Tr}^{\mathcal{K}\ell}\\
&\quad = \mathsf{Tr}^{\mathcal{K}\ell}\big(\,\mathsf{dis}^{-1} \bullet \mathsf{Tr}^{\mathcal{K}\ell}\big(\,\mathsf{dis} \bullet c \bullet \mathsf{dis}\,\big) \bullet \mathsf{dis}\,\big) \bullet ((\lambda \circ \rho) \otimes \mathrm{id})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by naturality for } \mathsf{Tr}^{\mathcal{K}\ell}\\
&\quad = \mathsf{Tr}(\mathsf{Tr}(c)) \bullet ((\lambda \circ \rho) \otimes \mathrm{id}).
\end{aligned}
$$

Superposing

The relevant diagram for components is

$$
\begin{tikzcd}
(D,E) \times (A+C, B+C) \arrow[r, "\Box"] \arrow[d, "\mathrm{id} \times \mathsf{Tr}"'] & (D+(A+C), E+(B+C)) \arrow[d, "{\langle \mathsf{arr}(\alpha_+^{-1}), \mathrm{id}, \mathsf{arr}(\alpha_+) \rangle}"] \\
(D,E) \times (A,B) \arrow[d, "\Box"'] & \begin{array}{c} ((D+A)+C, D+(A+C)) \times \\ (D+(A+C), E+(B+C)) \times \\ (E+(B+C), (E+B)+C) \end{array} \arrow[d, "(\ggg \times \mathrm{id}) \circ \ggg"] \\
(D+A, E+B) & ((D+A)+C, (E+B)+C) \arrow[l, "\mathsf{Tr}"]
\end{tikzcd}
$$

Pictorially, we have



For coalgebraic components

$$Y \otimes E \xrightarrow{d} Y \otimes D$$

$$X \otimes (A + C) \xrightarrow{c} X \otimes (B + C),$$

this involves an isomorphism of components:

$$\begin{tikzcd}
T(((I \otimes (Y \otimes X)) \otimes I) \otimes (E + B)) \arrow[r, "T((\lambda\circ\rho)\otimes\mathrm{id})", "\cong"'] & T((Y \otimes X) \otimes (E + B)) \\
((I \otimes (Y \otimes X)) \otimes I) \otimes (E + B) \arrow[u, "\mathsf{Tr}((\mathsf{arr}(\alpha_+^{-1}) \ggg (d \Box c)) \ggg \mathsf{arr}(\alpha_+))"] \arrow[r, "\cong", "(\lambda\circ\rho)\otimes\mathrm{id}"'] & (Y \otimes X) \otimes (E + B) \arrow[u, "d \Box \mathsf{Tr}(c)"']
\end{tikzcd}$$

We then proceed along what should by now be familiar lines.

7. Traced monoidal category of resumptions

In this section we use the previous results for operators and equations on components to prove that the category of T-*resumptions* is traced symmetric monoidal. This general result holds for a large class of monads T with suitable assumptions, and thus generalises the result in Abramsky *et al.* (2002) that focuses on the lift monad $T = 1 + (-)$. Although we do not show it fully, the technical development is an instance of the theory, which was developed in Hasuo *et al.* (2008) and Hasuo *et al.* (2009), on the *microcosm principle* (Baez and Dolan 1998). The application of this general theory exploits a characterisation of resumptions as elements of a final component.

Throughout Section 7, the base category $\mathbb{C}$ is fixed to be **Set**, the category of sets and functions. It is a symmetric monoidal closed category with Cartesian product $\times$ as tensor $\otimes$ and the singleton 1 as monoidal unit I, and is also equipped with distributive coproducts $+, 0$. All the results in the previous sections are valid in this base category.

7.1. *Resumptions*

The notion of a *resumption* was introduced in Milner (1975) to help provide a denotational semantics for interactive computing agents. A historical account is given in Abramsky *et al.* (2002, Section 5.4.1), and our recap of it here is adapted for the current context.

First, we consider a component $X \times A \xrightarrow{c} X \times B$, a map in **Set**. It is a component (3) where $T = \mathrm{id}$ is the trivial monad and $\otimes$ is chosen to be Cartesian product $\times$. It belongs to the category $\mathbf{Comp}(\mathrm{id}, A, B)$; with $T = \mathrm{id}$ this component does not exhibit any effect in its execution. In the theory of automata, such a machine is called a *Mealy machine*; it can also be thought of as a (simple version of a) *transducer*. The task that such a machine is expected to perform is the transformation of (infinite) A-streams into B-streams, and the transformation should be performed letter by letter.

A *resumption* is an extensional view of the behaviour of such a machine. Specifically, the above machine c induces a resumption formalised as a stream function $r : A^\omega \to B^\omega$

that is *causal*, meaning that the nth letter of the output stream only depends on the first n letters of the input†.

We will now take a coalgebraic view of components and resumptions. A component $X \times A \xrightarrow{c} X \times B$ is the same thing as a map $X \to (X \times B)^A$, and hence is a coalgebra for the functor $(- \times B)^A$. As noted in Abramsky *et al.* (2002), the 'behaviours by coinduction' paradigm in the theory of coalgebra (Jacobs and Rutten 1997; Rutten 2000) is also valid in this setting. Namely, the set $Z_{A,B}$ of resumptions (that is, causal stream functions) carries a canonical $(- \times B)^A$-coalgebra structure:

$$\begin{array}{ccc} Z_{A,B} & \xrightarrow[\cong]{\zeta_{A,B}} & (Z_{A,B} \times B)^A \\ \big(r : A^\omega \to B^\omega,\ \text{causal}\big) & \longmapsto & \lambda a. \Big(\big(\lambda \overrightarrow{a'}.\, \mathsf{tail}(r(a \cdot \overrightarrow{a'}))\big),\ \mathsf{head}(r(a \cdot \overrightarrow{a}))\Big). \end{array}$$

Here $a \cdot \overrightarrow{a}$ is a letter $a \in A$ followed by an arbitrary stream $\overrightarrow{a}$; the value of $\mathsf{head}(r(a \cdot \overrightarrow{a}))$ does not depend on $\overrightarrow{a}$ since r is causal. Moreover, it is a standard result that this coalgebra $\zeta_{A,B}$ is a final one. Given an arbitrary component $c : X \times A \to X \times B$ (that is, a coalgebra $c : X \to (X \times B)^A$), finality of $\zeta_{A,B}$ induces the *behaviour map*

$$\begin{tikzcd} (X \times B)^A \arrow[r, dashed] & (Z_{A,B} \times B)^A \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}(c)"'] & Z_{A,B} \arrow[u, "\text{final}"', "\cong"] \end{tikzcd}$$

which carries a state $x \in X$ to the behaviour $\mathsf{beh}(c)(x)$ of c, in the case of execution with x as the initial state, represented by a resumption. To summarise, the set of resumptions from A to B carries a final $(- \times B)^A$ coalgebra.

We have restricted ourselves to the trivial monad $T = \mathrm{id}$ for the purpose of illustration of resumptions. However, this choice of a monad T does not satisfy the assumption in Section 6 for $\mathcal{K}\ell(T)$ to be traced: an iteration of a total function can fail to be total because of an infinite loop. For monads T in general, especially those satisfying the assumption in Section 6, we generalise the above characterisation of resumptions as follows – this was also done in Abramsky *et al.* (2002).

Definition 7.1 (T-resumption). Let T be a monad on **Set** such that for any sets A and B, a final $(T(- \times B))^A$-coalgebra

$$\zeta^T_{A,B} \ : \ Z^T_{A,B} \xrightarrow{\cong} \big(T(Z^T_{A,B} \times B) \big)^A \qquad \text{in } \mathbf{Set}$$

exists. A T-*resumption* from A to B is an element of the carrier $Z^T_{A,B}$.

Morphisms of $T(- \times B)^A$-coalgebras are precisely morphisms of components: there is an isomorphism of categories

$$\mathbf{Coalg}(T(- \times B)^A) \cong \mathbf{Comp}(T, A, B).$$

† This is how they are formalised in Rutten (2006). Equivalent formulations are as string functions $A^* \to B^*$ that are length preserving and prefix closed (Pattinson 2003), and as functions $A^+ \to B$ where A^+ is the set of (finite-length) strings of length $\geqslant 1$.

Hence T-resumptions form the state space of a *final component*.

Assumption 7.2. For the rest of this section we assume that a monad T on **Set** satisfies both the assumption in Section 6 (namely, Jacobs (2010, Requirements 4.7)) as well as the one in Definition 7.1. The former consists of $\mathcal{K}\ell(T)$ being $\mathbf{Dcpo}_{\perp}$-enriched, T being 'semi-additive', and so on. This ensures that we have a trace operator $\mathsf{Tr}^{\mathcal{K}\ell}$, which enables us to capture resumptions by a final coalgebra.

Such monads include the lift monad $1+(-)$, the κ-bounded powerset monad $\mathcal{P}^{<\kappa}$ with an uncountable weakly inaccessible cardinal κ and the (discrete) subdistribution monad $\mathcal{D}$. With regard to the monad $\mathcal{P}^{<\kappa}$, the cardinal κ must be larger than $\aleph_0$ so that an increasing ω-sequence in the set $\mathcal{P}^{<\kappa}(X)$ has its supremum inside $\mathcal{P}^{<\kappa}(X)$ – this is required for the trace assumption in Section 6. At the same time, κ is assumed to be weakly inaccessible so that Barr's final coalgebra theorem (Barr 1993) ensures the existence of final coalgebras. Such an explicit size restriction is not needed for the subdistribution monad $\mathcal{D}$, since the condition $\sum_x d(x) \leqslant 1$ implies that the support $\{x \in X \mid d(x) \neq 0\}$ is at most countable – see, for example, Sokolova (2005, Proposition 2.1.2).

It is generally hard to describe what a T-resumption looks like concretely. It is a tree, much like a *synchronisation tree* (Milner 1980), but its depth and branching degree are very often larger than $\aleph_0$. A tractable description is possible for the lift monad: much like for the identity monad, it is represented by a function $r : A^\omega \to B^* + B^\omega$ with a suitably generalised causality requirement.

7.2. *The microcosm principle*

One can form the *category of T-resumptions* by arranging T-resumptions as morphisms in the category.

Definition 7.3 (The category T-Res). For a monad T satisfying Assumption 7.2, we define the *category of T-resumptions*, denoted by T-**Res**, by the following data:

— An object A of T-**Res** is a set $A \in \mathbf{Set}$.
— An arrow $r : A \to B$ in T-**Res** is a T-resumption from A to B (*cf.* Definition 7.1). So we have $\mathrm{Hom}_{T\text{-}\mathbf{Res}}(A, B) = Z^T_{A,B}$.

Its actual structure as a category (the composition and identity) will be described shortly.

The main point of Abramsky *et al.* (2002, Section 5.4) is that the category of resumptions $(1+(-))$-**Res** for the lift monad is symmetric traced monoidal, and that it gives rise to a compact closed category of (stateful) games (Abramsky and Jagadeesan 1994) after applying the Int-construction (Joyal *et al.* 1996). Its generalisation to a wider class of monads T (other than $T = 1+(-)$) is one of our main technical contributions.

Theorem 7.4. For a monad T satisfying Assumption 7.2, the category T-**Res** of T-resumptions is symmetric traced monoidal.

This result, in fact, is an immediate corollary of what we have already observed: the traced monoidal structure of T-**Res** follows from the composition and trace operators for components introduced in Sections 4 and 6.

To illustrate the situation, consider arranging components, instead of resumptions, as morphisms from A to B. Between such components c and d with the same input/output types, we possibly have a morphism of components, see (4), and this 'morphism' f can be drawn between components, as follows:

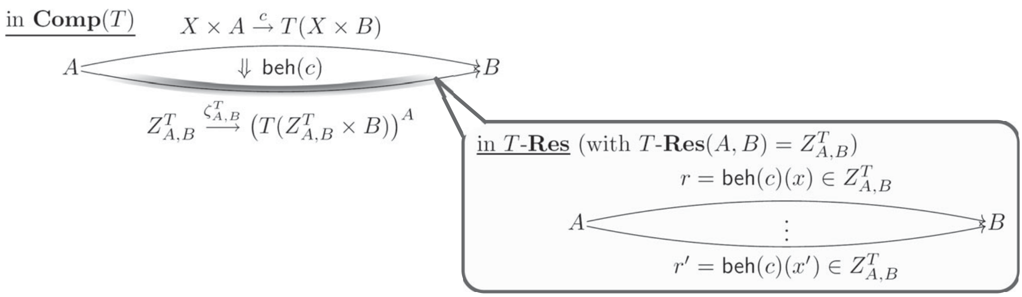
$$\begin{tikzcd} A \arrow[r, bend left, "c"] \arrow[r, bend right, "d"'] \arrow[r, phantom, "\Downarrow f"] & B \end{tikzcd}$$

This motivates a 2-categorical approach to components. The 2-category we have in mind has sets as 0-cells, components as 1-cells and morphisms of components as 2-cells. It would be natural for us to introduce (horizontal) composition of 1-cells as the sequential composition $\ggg$ of components (Section 4) and the identity 1-cell $A \to A$ as the one-state component $\mathsf{arr}(\mathrm{id}_A)$ from (5).

However, Lemma 4.2 (2)–(3) indicates that such horizontal composition of 1-cells satisfies the unit law and associativity only up-to canonical isomorphisms. This means the resulting structure is a *bicategory* and not a 2-category – see, for example, Borceux (1994). This bicategory is much like the one in the bicategorical approach to processes (Katis *et al.* 1997), and we will denote it by $\mathbf{Comp}(T)$. This extends our previous notation since its hom-category from A to B is given by the category $\mathbf{Comp}(T, A, B)$ of components.

What we showed in Section 6 is, essentially, that the bicategory $\mathbf{Comp}(T)$ is equipped with traced monoidal structure†. Its underlying monoidal structure is given by additive parallel composition $\Box$ (Section 4.2); in particular, it is a binary coproduct $+$ on objects.

The way we look at the category T-**Res** of resumptions is as a 'thin slice' of the bicategory $\mathbf{Comp}(T)$ of components. The two have the same family of objects, the former's homset T-$\mathbf{Res}(A, B)$ resides in the latter's $\mathbf{Comp}(T, A, B)$ as (the carrier of) a 1-cell and T-$\mathbf{Res}(A, B)$ is still 'behaviourally universal' through its finality.



We can then derive the structure of T-**Res** as a traced monoidal category from the corresponding 'outer' structure of $\mathbf{Comp}(T)$ – this follows from the general theory previously developed in Hasuo *et al.* (2008) and Hasuo *et al.* (2009). The general theory identifies the situation as an instance of the *microcosm principle* (Baez and Dolan 1998). The latter refers to a situation where 'an algebra resides in another algebra, both for the

† An axiomatisation of the notion of 'traced monoidal bicategory' would involve delicate coherence conditions, but we do not aim for such a general axiomatisation as we will focus on one specific instance.

same algebraic specification', a prototypical example of which is a monoid in a monoidal category (Mac Lane 1998). As a result, the homsets of T-**Res** form a traced monoidal category, residing in the hom-categories of **Comp**(T) that form a 'traced monoidal bicategory' – see Hasuo *et al.* (2009) for details of the generic situation. Rather than fully laying out the general theory, however, we shall now describe a concrete instantiation adapted to the current setting.

7.3. *Resumptions form a category*

Notation 7.5. The functor $\big(T(- \times B)\big)^A$, for which a coalgebra is a component with A-input and B-output, is denoted by $F_{A,B}$. The monad T is fixed throughout the rest of this discussion, so it will be suppressed, and we will write $Z_{A,B}$ rather than $Z^T_{A,B}$ for the homset of T-resumptions.

We will first derive the sequential composition operator $\circ^{T\text{-}\mathbf{Res}}$ that acts on resumptions, which is obtained from the way we compose arrows in T-**Res**. The following coinduction diagram in **Set** defines the operator:

$$\begin{tikzcd}
F_{A,C}(Z_{A,B} \times Z_{B,C}) \arrow[r, dashed] & F_{A,C}(Z_{A,C}) \\
Z_{A,B} \times Z_{B,C} \arrow[u, "\zeta_{A,B} \ggg \zeta_{B,C}"] \arrow[r, dashed, "\circ^{T\text{-}\mathbf{Res}}_{A,B,C}"'] & Z_{A,C} \arrow[u, "\text{final}"', "\zeta_{A,C}"]
\end{tikzcd} \tag{22}$$

Recall that $\zeta_{A,B}$ is a final $F_{A,B}$-coalgebra (Definition 7.1). The sequential composition of $\zeta_{B,C}$ after $\zeta_{A,B}$ yields the component shown on the left, which is an $F_{A,C}$-coalgebra with a state space $Z_{A,B} \times Z_{B,C}$. It then induces a unique map into the final $F_{A,C}$-coalgebra, as in the above diagram. Thus, we have obtained a function

$$\circ^{T\text{-}\mathbf{Res}}_{A,B,C} : Z_{A,B} \times Z_{B,C} \longrightarrow Z_{C,A},$$

that is,

$$\mathrm{Hom}_{T\text{-}\mathbf{Res}}(A,B) \times \mathrm{Hom}_{T\text{-}\mathbf{Res}}(B,C) \longrightarrow \mathrm{Hom}_{T\text{-}\mathbf{Res}}(A,C).$$

Similarly, the identity morphism $\mathrm{id}^{T\text{-}\mathbf{Res}}_A$ in T-**Res** is derived by coinduction from the one-state component $\mathsf{arr}(\mathrm{id}_A)$:

$$\begin{tikzcd}
F_{A,A}(1) \arrow[r, dashed] & F_{A,A}(Z_{A,A}) \\
1 \arrow[u, "\mathsf{arr}(\mathrm{id}_A)"] \arrow[r, dashed, "\mathrm{id}^{T\text{-}\mathbf{Res}}_A"'] & Z_{A,A} \arrow[u, "\text{final}"', "\zeta_{A,A}"]
\end{tikzcd} \tag{23}$$

We will now prove the associativity of $\circ^{T\text{-}\mathbf{Res}}$. The emphasis here is on the fact that, through coinduction, the goal is essentially reduced to associativity (Lemma 4.2(3)) of $\ggg$, which is the corresponding 'outer' operator.

Specifically, by diagram (22), the map

$$\circ^{T\text{-}\mathbf{Res}}_{A,B,C} : Z_{A,B} \times Z_{B,C} \longrightarrow Z_{C,A}$$

is a component morphism from $\zeta_{A,B} \ggg \zeta_{B,C}$ to $\zeta_{A,C}$. Hence, by the functoriality of $\ggg$ (Lemma 4.2 (4)), we obtain a component morphism

$$\circ^{T\text{-}\mathbf{Res}}_{A,B,C} \times \mathrm{id}_{Z_{C,D}} \;:\; (\zeta_{A,B} \ggg \zeta_{B,C}) \ggg \zeta_{C,D} \longrightarrow \zeta_{A,C} \ggg \zeta_{C,D}.$$

This means that the left-hand square in the following diagram commutes, and the right-hand square also commutes since it is just diagram (22) defining $\circ^{T\text{-}\mathbf{Res}}_{A,C,D}$:

$$\begin{tikzcd}
F_{A,D}((Z_{A,B} \times Z_{B,C}) \times Z_{C,D}) \arrow[r] & F_{A,D}(Z_{A,C} \times Z_{C,D}) \arrow[r, dashed] & F_{A,D}(Z_{A,D}) \\
(Z_{A,B} \times Z_{B,C}) \times Z_{C,D} \arrow[u, "(\zeta_{A,B} \ggg \zeta_{B,C}) \ggg \zeta_{C,D}"] \arrow[r, "\circ^{T\text{-}\mathbf{Res}}_{A,B,C} \times Z_{C,D}"'] & Z_{A,C} \times Z_{C,D} \arrow[u, "\zeta_{A,C} \ggg \zeta_{C,D}"] \arrow[r, dashed, "\circ^{T\text{-}\mathbf{Res}}_{A,C,D}"'] & Z_{A,D} \arrow[u, "\mathrm{final}"', "\zeta_{A,D}"]
\end{tikzcd}$$

The next diagram in **Set** commutes for the same reasons – the top square commutes by associativity of $\ggg$ (Lemma 4.2 (3)):

$$\begin{tikzcd}
(Z_{A,B} \times Z_{B,C}) \times Z_{C,D} \arrow[r, "(\zeta_{A,B} \ggg \zeta_{B,C}) \ggg \zeta_{C,D}"] \arrow[d, "\alpha^{-1}"', "\cong"] & F_{A,D}((Z_{A,B} \times Z_{B,C}) \times Z_{C,D}) \arrow[d] \\
Z_{A,B} \times (Z_{B,C} \times Z_{C,D}) \arrow[r, "\zeta_{A,B} \ggg (\zeta_{B,C} \ggg \zeta_{C,D})"] \arrow[d, "Z_{A,B} \times \circ^{T\text{-}\mathbf{Res}}_{B,C,D}"'] & F_{A,D}(Z_{A,B} \times (Z_{B,C} \times Z_{C,D})) \arrow[d] \\
Z_{A,B} \times Z_{B,D} \arrow[r, "\zeta_{A,B} \ggg \zeta_{B,D}"] \arrow[d, dashed, "\circ^{T\text{-}\mathbf{Res}}_{A,B,D}"'] & F_{A,D}(Z_{A,B} \times Z_{B,D}) \arrow[d, dashed] \\
Z_{A,D} \arrow[r, "\mathrm{final}", "\zeta_{A,D}"'] & F_{A,D}(Z_{A,D})
\end{tikzcd}$$

(in this diagram, unlike our convention elsewhere, and purely for typesetting convenience, the coalgebras $X \to FX$ are written horizontally instead of vertically).

We can now conclude that the following diagram commutes, since the previous two diagrams show that the two composites are both coalgebra morphisms from $(\zeta_{A,B} \ggg \zeta_{B,C}) \ggg \zeta_{C,D}$ to a final coalgebra $\zeta_{A,D}$:

$$\begin{tikzcd}
(Z_{A,B} \times Z_{B,C}) \times Z_{C,D} \arrow[rr, "\circ^{T\text{-}\mathbf{Res}}_{A,B,C} \times Z_{C,D}"] \arrow[d, "\alpha^{-1}"'] & & Z_{A,C} \times Z_{C,D} \arrow[d, "\circ^{T\text{-}\mathbf{Res}}_{A,C,D}"] \\
Z_{A,B} \times (Z_{B,C} \times Z_{C,D}) \arrow[r, "Z_{A,B} \times \circ^{T\text{-}\mathbf{Res}}_{B,C,D}"'] & Z_{A,B} \times Z_{B,D} \arrow[r, "\circ^{T\text{-}\mathbf{Res}}_{A,B,D}"'] & Z_{A,D}
\end{tikzcd}$$

This gives the associativity of $\circ^{T\text{-}\mathbf{Res}}$.

The left- and right-unit laws for T-**Res** amount to the following diagram:

$$\begin{tikzcd}
Z_{A,B} \arrow[r, "\cong"] \arrow[d, "\cong"'] \arrow[rrd, equal] & 1 \times Z_{A,B} \arrow[r, "\mathrm{id}^{T\text{-}\mathbf{Res}}_{A} \times Z_{A,B}"] & Z_{A,A} \times Z_{A,B} \arrow[d, "\circ^{T\text{-}\mathbf{Res}}_{A,A,B}"] \\
Z_{A,B} \times 1 \arrow[r, "Z_{A,B} \times \mathrm{id}^{T\text{-}\mathbf{Res}}_{B}"'] & Z_{A,B} \times Z_{B,B} \arrow[r, "\circ^{T\text{-}\mathbf{Res}}_{A,B,B}"'] & Z_{A,B}
\end{tikzcd}$$

This diagram commutes, essentially, because of the unit laws (Lemma 4.2 (2)) for the outer operators $\ggg$ and $\mathsf{arr}(\mathrm{id})$, much like in the case for associativity.

The following proposition summarises what we have achieved so far.

Proposition 7.6. The data T-**Res** in Definition 7.3 do indeed form a category, with composition of arrows given by $\circ^{T\text{-}\mathbf{Res}}$ in (22) and identity arrows by $\mathrm{id}^{T\text{-}\mathbf{Res}}$ in (23).

In fact, this proposition is just a special case of Krstić *et al.* (2001, Theorem 1), which is more general because it works for an axiomatically introduced class of functors $\{F_{A,B}\}_{A,B}$, which corresponds roughly to the notion of a *lax* $\mathbb{L}$*-functor* in Hasuo *et al.* (2008) and Hasuo *et al.* (2009), instead of our concrete description

$$F_{A,B} = \big(T(B \times -)\big)^{A}$$

(see Notation 7.5). However, we will now go beyond Krstić *et al.* (2001) by introducing a symmetric monoidal structure on T-**Res** and a trace operator on top of it.

7.4. *Resumptions carry symmetric monoidal structure*

Endowing the category T-**Res** with a traced monoidal structure follows along pretty much the same lines – we will describe the structure in some detail in the following.

The monoidal structure is given by *additive parallel composition* $\Box^{T\text{-}\mathbf{Res}}$ of resumptions, which is derived from $\Box$ on components (Section 4.2), the corresponding outer structure. It acts on objects as a sum of sets:

$$A \,\Box^{T\text{-}\mathbf{Res}}\, B \;:=\; A + B. \tag{24}$$

On arrows, its action

$$\begin{aligned} \Box^{T\text{-}\mathbf{Res}}_{A,C,B,D} \;:\; Z_{A,B} \times Z_{C,D} \longrightarrow Z_{A+C,B+D}, \quad &\text{that is} \\ \mathrm{Hom}_{T\text{-}\mathbf{Res}}(A,B) \times \mathrm{Hom}_{T\text{-}\mathbf{Res}}(C,D) &\longrightarrow \mathrm{Hom}_{T\text{-}\mathbf{Res}}(A+C,B+D) \end{aligned}$$

is induced through the following diagram, which is similar to (22):

$$\begin{tikzcd} F_{A+C,B+D}(Z_{A,B} \times Z_{C,D}) \arrow[r, dashed] & F_{A+C,B+D}(Z_{A+C,B+D}) \\ Z_{A,B} \times Z_{C,D} \arrow[u, "\zeta_{A,B}\,\Box\,\zeta_{C,D}"] \arrow[r, dashed, "\Box^{T\text{-}\mathbf{Res}}_{A,C,B,D}"'] & Z_{A+C,B+D} \arrow[u, "\text{final}"', "\zeta_{A+C,B+D}"] \end{tikzcd} \tag{25}$$

Lemma 7.7. The mapping $\Box^{T\text{-}\mathbf{Res}}$ yields a functor $\Box^{T\text{-}\mathbf{Res}} : T\text{-}\mathbf{Res} \times T\text{-}\mathbf{Res} \to T\text{-}\mathbf{Res}$.

Proof. We will first prove the preservation of identities. We have the following successive morphisms of coalgebraic components, where coalgebras $X \to FX$ are written horizontally:

$$\begin{tikzcd}
1 \arrow[r, "\mathsf{arr}(\mathrm{id}_{A+B})"] \arrow[d, "\cong"'] & F_{A+B,A+B}(1) \arrow[d, "\cong"] \\
1 \times 1 \arrow[r, "\mathsf{arr}(\mathrm{id}_A)\Box\mathsf{arr}(\mathrm{id}_B)"] \arrow[d, "\mathrm{id}_A^{T\text{-}\mathbf{Res}}\times\mathrm{id}_B^{T\text{-}\mathbf{Res}}"'] & F_{A+B,A+B}(1 \times 1) \arrow[d] \\
Z_{A,A} \times Z_{B,B} \arrow[r, "\zeta_{A,A}\Box\zeta_{B,B}"] \arrow[d, "\Box^{T\text{-}\mathbf{Res}}_{A,B,A,B}"'] & F_{A+B,A+B}(Z_{A,A} \times Z_{B,B}) \arrow[d] \\
Z_{A+B,A+B} \arrow[r, "\text{final}", "\zeta_{A+B,A+B}"'] & F_{A+B,A+B}(Z_{A+B,A+B})
\end{tikzcd}$$

The first square commutes because of the compatibility of $\Box$ and arr (Lemma 4.9 (1)). The second square commutes because of the definition of $\mathrm{id}^{T\text{-}\mathbf{Res}}$ (23) and functoriality of $\Box$ (see the discussion following Definition 4.7). And the third square is the definition of $\Box^{T\text{-}\mathbf{Res}}$ (25). This proves that the following diagram commutes, since the two composites are both morphisms from the component $\mathsf{arr}(\mathrm{id}_{A+B})$ to the final $\zeta_{A+B,A+B}$:

$$\begin{tikzcd}
1 \arrow[r, "\cong"] \arrow[drr, "\mathrm{id}^{T\text{-}\mathbf{Res}}_{A+B}"'] & 1 \times 1 \arrow[r, "\mathrm{id}_A^{T\text{-}\mathbf{Res}}\times\mathrm{id}_B^{T\text{-}\mathbf{Res}}"] & Z_{A,A} \times Z_{B,B} \arrow[d, "\Box^{T\text{-}\mathbf{Res}}_{A,B,A,B}"] \\
& & Z_{A+B,A+B}
\end{tikzcd}$$

Hence $\Box^{T\text{-}\mathbf{Res}} : T\text{-}\mathbf{Res} \times T\text{-}\mathbf{Res} \to T\text{-}\mathbf{Res}$ preserves identities.

We turn now to the preservation of composition, and proceed using similar arguments. We have the following two (parallel) series of morphisms of coalgebraic components, which are all arrows in the category $\mathbf{Comp}(T, A + B, A'' + B'')$:

$$\begin{array}{rl}
& (\zeta_{A,A'} \ggg \zeta_{A',A''}) \Box (\zeta_{B,B'} \ggg \zeta_{B',B''}) \\
\xrightarrow{\qquad\cong\qquad} & (\zeta_{A,A'} \Box \zeta_{B,B'}) \ggg (\zeta_{A',A''} \Box \zeta_{B',B''}) \\
\xrightarrow{(\Box^{T\text{-}\mathbf{Res}}_{A,B,A',B'})\times(\Box^{T\text{-}\mathbf{Res}}_{A',B',A'',B''})} & \zeta_{A+B,A'+B'} \ggg \zeta_{A'+B',A''+B''} \\
\xrightarrow{\circ^{T\text{-}\mathbf{Res}}_{A+B,A'+B',A''+B''}} & \zeta_{A+B,A''+B''}\,;
\end{array}$$

and

$$\begin{array}{rl}
& (\zeta_{A,A'} \ggg \zeta_{A',A''}) \Box (\zeta_{B,B'} \ggg \zeta_{B',B''}) \\
\xrightarrow{(\circ^{T\text{-}\mathbf{Res}}_{A,A',A''})\times(\circ^{T\text{-}\mathbf{Res}}_{B,B',B''})} & \zeta_{A,A''} \Box \zeta_{B,B''} \\
\xrightarrow{\Box^{T\text{-}\mathbf{Res}}_{A,B,A'',B''}} & \zeta_{A+B,A''+B''}\,.
\end{array}$$

The first (isomorphism) arrow follows from Lemma 4.9 (2). The second is a morphism of components from the definition of $\Box^{T\text{-}\mathbf{Res}}$ and the functoriality of $\ggg$ (Lemma 4.2 (4)). And the third is just the definition of $\circ^{T\text{-}\mathbf{Res}}$. The remaining two are component morphisms for similar reasons. Since the coalgebraic component $\zeta_{A+B,A''+B''}$ is a final coalgebra, we conclude that the two composites above are identical. In particular, by taking their underlying functions, we have the commuting diagram

$$\begin{tikzcd}[column sep=small]
(Z_{A,A'} \times Z_{A',A''}) \times (Z_{B,B'} \times Z_{B',B''}) \arrow[rr, "\cong"] \arrow[d, "\circ^{T\text{-}\mathbf{Res}}_{A,A',A''} \times \circ^{T\text{-}\mathbf{Res}}_{B,B',B''}"'] & & (Z_{A,A'} \times Z_{B,B'}) \times (Z_{A',A''} \times Z_{B',B''}) \arrow[d, "\square^{T\text{-}\mathbf{Res}}_{A,B,A',B'} \times \square^{T\text{-}\mathbf{Res}}_{A',B',A'',B''}"] \\
Z_{A,A''} \times Z_{B,B''} \arrow[dr, "\square^{T\text{-}\mathbf{Res}}_{A,B,A'',B''}"'] & & Z_{A+B,A'+B'} \times Z_{A'+B',A''+B''} \arrow[dl, "\circ^{T\text{-}\mathbf{Res}}_{A+B,A'+B',A''+B''}"] \\
& Z_{A+B,A''+B''} &
\end{tikzcd}$$

in **Set**. Hence

$$\square^{T\text{-}\mathbf{Res}} : T\text{-}\mathbf{Res} \times T\text{-}\mathbf{Res} \to T\text{-}\mathbf{Res}$$

preserves composition of arrows. $\square$

The monoidal unit for $(T\text{-}\mathbf{Res}, \square^{T\text{-}\mathbf{Res}})$ is the empty set 0. We still need to describe associativity, unit and symmetry isomorphisms – they appear in the proof of the following result.

Proposition 7.8. $(T\text{-}\mathbf{Res}, \square^{T\text{-}\mathbf{Res}}, 0)$ is a symmetric monoidal category.

Proof. We shall first describe the definition of the structural isomorphisms. Then we prove that:

(1) they are indeed isomorphisms;
(2) they are natural; and
(3) they are coherent as in Mac Lane (1998).

An associativity isomorphism

$$\alpha^{T\text{-}\mathbf{Res}} : A + (B + C) \xrightarrow{\cong} (A + B) + C$$

in T-**Res** (recall that $\square^{T\text{-}\mathbf{Res}}$ is $+$ on objects, see (24)) is induced by the diagram

$$\begin{tikzcd}
F_{A+(B+C),(A+B)+C}(1) \arrow[r, dashed] & F_{A+(B+C),(A+B)+C}(Z_{A+(B+C),(A+B)+C}) \\
1 \arrow[u, "\mathsf{arr}(\alpha_+)"] \arrow[r, dashed, "\alpha^{T\text{-}\mathbf{Res}}_{A,B,C}"'] & Z_{A+(B+C),(A+B)+C} \arrow[u, "\text{final}"', "\zeta_{A+(B+C),(A+B)+C}"]
\end{tikzcd}$$

Here, α_+ on the left, in $\mathsf{arr}(\alpha_+)$, is the isomorphism

$$A + (B + C) \xrightarrow{\cong} (A + B) + C$$

in **Set**. In exactly the same way, we obtain unit isomorphisms $\lambda^{T\text{-}\mathbf{Res}}$ and $\rho^{T\text{-}\mathbf{Res}}$ and symmetry isomorphisms $\gamma^{T\text{-}\mathbf{Res}}$ from the corresponding isomorphisms for $+$ in **Set**.

It is easy to see that all these are indeed isomorphisms – we will just write down the proof for $\alpha^{T\text{-}\mathbf{Res}}$. Let $\overline{\alpha}^{T\text{-}\mathbf{Res}}_{A,B,C}$ be the following resumption induced by the isomorphism

$$\alpha_+^{-1} : (A + B) + C \xrightarrow{\cong} A + (B + C)$$

in **Set**:

$$\begin{tikzcd}
F_{(A+B)+C,A+(B+C)}(1) \arrow[rrr, dashed] & & & F_{(A+B)+C,A+(B+C)}(Z_{(A+B)+C,A+(B+C)}) \\
1 \arrow[u, "\mathsf{arr}(\alpha_+^{-1})"] \arrow[rrr, dashed, "\overline{\alpha}^{T\text{-}\mathbf{Res}}_{A,B,C}"'] & & & Z_{(A+B)+C,A+(B+C)} \arrow[u, "\mathsf{final}"', "\zeta_{(A+B)+C,A+(B+C)}"]
\end{tikzcd}$$

We claim that this $\overline{\alpha}^{T\text{-}\mathbf{Res}}_{A,B,C}$ is the inverse of $\alpha^{T\text{-}\mathbf{Res}}_{A,B,C}$, that is,

$$\begin{tikzcd}
1 \times 1 \arrow[d, "\overline{\alpha}^{T\text{-}\mathbf{Res}}_{A,B,C} \times \alpha^{T\text{-}\mathbf{Res}}_{A,B,C}"'] & 1 \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[ddl, bend right, "\mathrm{id}^{T\text{-}\mathbf{Res}}_{(A+B)+C}"] \arrow[ddr, bend left, "\mathrm{id}^{T\text{-}\mathbf{Res}}_{A+(B+C)}"'] & 1 \times 1 \arrow[d, "\alpha^{T\text{-}\mathbf{Res}}_{A,B,C} \times \overline{\alpha}^{T\text{-}\mathbf{Res}}_{A,B,C}"] \\
Z_{(A+B)+C,A+(B+C)} \times Z_{A+(B+C),(A+B)+C} \arrow[d, "\circ^{T\text{-}\mathbf{Res}}"'] & & Z_{A+(B+C),(A+B)+C} \times Z_{(A+B)+C,A+(B+C)} \arrow[d, "\circ^{T\text{-}\mathbf{Res}}"] \\
Z_{(A+B)+C,(A+B)+C} & & Z_{A+(B+C),A+(B+C)}
\end{tikzcd}$$

We will prove commutativity of the triangle on the right; the proof for the other is similar. As before, we prove that the following arrows are all component morphisms leading to a final coalgebra $\zeta_{A+(B+C),A+(B+C)}$:

$$\begin{aligned}
& (\mathsf{arr}(\mathrm{id}_{A+(B+C)})) \\
\xrightarrow{\cong}\ & \mathsf{arr}((\alpha_+)_{A,B,C}) \ggg \mathsf{arr}((\alpha_+^{-1})_{A,B,C}) \\
\xrightarrow{\alpha^{T\text{-}\mathbf{Res}}_{A,B,C} \times \overline{\alpha}^{T\text{-}\mathbf{Res}}_{A,B,C}}\ & \zeta_{A+(B+C),(A+B)+C} \ggg \zeta_{(A+B)+C,A+(B+C)} \\
\xrightarrow{\circ^{T\text{-}\mathbf{Res}}_{A+(B+C),(A+B)+C,A+(B+C)}}\ & \zeta_{A+(B+C),A+(B+C)};
\end{aligned}$$

and

$$\begin{aligned}
& (\mathsf{arr}(\mathrm{id}_{A+(B+C)})) \\
\xrightarrow{\mathrm{id}^{T\text{-}\mathbf{Res}}_{A+(B+C)}}\ & \zeta_{A+(B+C),A+(B+C)}.
\end{aligned}$$

The first (isomorphism) arrow follows from Lemma 4.2 (1). The second is from the definition of $\alpha^{T\text{-}\mathbf{Res}}, \overline{\alpha}^{T\text{-}\mathbf{Res}}$ and the functoriality of $\ggg$ (Lemma 4.2 (4)). The third is the definition of $\circ^{T\text{-}\mathbf{Res}}$, and the last is the definition of $\mathrm{id}^{T\text{-}\mathbf{Res}}$. Thus we have proved that $\alpha^{T\text{-}\mathbf{Res}}_{A,B,C}$ is indeed an isomorphism.

We turn now to the naturality of $\alpha^{T\text{-}\mathbf{Res}}, \lambda^{T\text{-}\mathbf{Res}}, \rho^{T\text{-}\mathbf{Res}}$ and $\gamma^{T\text{-}\mathbf{Res}}$, and again we will only present the proof for $\alpha^{T\text{-}\mathbf{Res}}$. This means we need to show the commutativity of

$$\begin{tikzcd}
A + (B + C) \arrow[r, "\alpha^{T\text{-}\mathbf{Res}}_{A,B,C}"] \arrow[d, "r \square^{T\text{-}\mathbf{Res}} (s \square^{T\text{-}\mathbf{Res}} t)"'] & (A + B) + C \arrow[d, "(r \square^{T\text{-}\mathbf{Res}} s) \square^{T\text{-}\mathbf{Res}} t"] \\
A' + (B' + C') \arrow[r, "\alpha^{T\text{-}\mathbf{Res}}_{A',B',C'}"'] & (A' + B') + C'
\end{tikzcd} \quad \text{in } T\text{-}\mathbf{Res},$$

for any resumptions r, s and t of suitable types, which amounts to the following diagram in **Set**:

$$\begin{tikzcd}
1 \times ((Z_{A,A'} \times Z_{B,B'}) \times Z_{C,C'}) \arrow[rr, "\cong"] \arrow[d, "\alpha^{T\text{-}\mathbf{Res}}_{A,B,C} \times \left(\Box^{T\text{-}\mathbf{Res}} \circ (\Box^{T\text{-}\mathbf{Res}} \times Z_{C,C'})\right)"'] & & (Z_{A,A'} \times (Z_{B,B'} \times Z_{C,C'})) \times 1 \arrow[d, "\left(\Box^{T\text{-}\mathbf{Res}} \circ (Z_{A,A'} \times \Box^{T\text{-}\mathbf{Res}})\right) \times \alpha^{T\text{-}\mathbf{Res}}_{A',B',C'}"] \\
Z_{A+(B+C),(A+B)+C} \times Z_{(A+B)+C,(A'+B')+C'} \arrow[dr, "\circ^{T\text{-}\mathbf{Res}}"'] & & Z_{A+(B+C),A'+(B'+C')} \times Z_{A'+(B'+C'),(A'+B')+C'} \arrow[dl, "\circ^{T\text{-}\mathbf{Res}}"] \\
& Z_{A+(B+C),(A'+B')+C'} &
\end{tikzcd}$$

Once again, this is achieved by showing that the above two composites are parallel coalgebra morphisms leading to a final coalgebra. Namely,

$$\begin{aligned}
& \mathsf{arr}((\alpha_+)_{A,B,C}) \ggg \left((\zeta_{A,A'} \,\Box\, \zeta_{B,B'}) \,\Box\, \zeta_{C,C'} \right) \\
\xrightarrow{\cong} \; & \left(\zeta_{A,A'} \,\Box\, (\zeta_{B,B'} \,\Box\, \zeta_{C,C'}) \right) \ggg \mathsf{arr}((\alpha_+)_{A',B',C'}) \\
\xrightarrow{\left(\Box^{T\text{-}\mathbf{Res}} \circ (Z_{A,A'} \times \Box^{T\text{-}\mathbf{Res}})\right) \times \alpha^{T\text{-}\mathbf{Res}}_{A',B',C'}} \; & \zeta_{A+(B+C),A'+(B'+C')} \ggg \zeta_{A'+(B'+C'),(A'+B')+C'} \\
\xrightarrow{\circ^{T\text{-}\mathbf{Res}}_{A+(B+C),(A'+B')+C',A'+(B'+C')}} \; & \zeta_{A+(B+C),(A'+B')+C'} \,;
\end{aligned}$$

and

$$\begin{aligned}
& \mathsf{arr}((\alpha_+)_{A,B,C}) \ggg \left((\zeta_{A,A'} \,\Box\, \zeta_{B,B'}) \,\Box\, \zeta_{C,C'} \right) \\
\xrightarrow{\alpha^{T\text{-}\mathbf{Res}}_{A,B,C} \times \left(\Box^{T\text{-}\mathbf{Res}} \circ (\Box^{T\text{-}\mathbf{Res}} \times Z_{C,C'})\right)} \; & \zeta_{A+(B+C),(A+B)+C} \ggg \zeta_{(A+B)+C,(A'+B')+C'} \\
\xrightarrow{\circ^{T\text{-}\mathbf{Res}}_{A+(B+C),(A+B)+C,(A'+B')+C'}} \; & \zeta_{A+(B+C),(A'+B')+C'} \,.
\end{aligned}$$

The first arrow is an isomorphism of components from Lemma 4.9 (3); the others are similar to those encountered earlier in the section. This concludes the proof of the naturality of $\alpha^{T\text{-}\mathbf{Res}}$.

Finally, we need to check the standard coherence conditions for $\alpha^{T\text{-}\mathbf{Res}}$, $\lambda^{T\text{-}\mathbf{Res}}$, $\rho^{T\text{-}\mathbf{Res}}$ and $\gamma^{T\text{-}\mathbf{Res}}$ in a symmetric monoidal category, as described in Mac Lane (1998), for example. We will just prove

$$\begin{tikzcd}
A + (0 + B) \arrow[rr, "\alpha^{T\text{-}\mathbf{Res}}_{A,0,B}", "\cong"'] \arrow[dr, "\mathrm{id}^{T\text{-}\mathbf{Res}}_A \Box^{T\text{-}\mathbf{Res}} \lambda^{T\text{-}\mathbf{Res}}_B"'] & & (A + 0) + B \arrow[dl, "\rho^{T\text{-}\mathbf{Res}}_A \Box^{T\text{-}\mathbf{Res}} \mathrm{id}^{T\text{-}\mathbf{Res}}_B"] \\
& A + B &
\end{tikzcd} \tag{26}$$

in T-**Res**; the other cases are similar. The above diagram amounts to the following diagram in **Set**:

$$\begin{tikzcd}
1 \arrow[rr, "\cong"] \arrow[d, "\cong"'] & & 1 \times 1 \arrow[d, "\mathrm{id}_A^{T\text{-}\mathbf{Res}} \times \lambda_B^{T\text{-}\mathbf{Res}}"] \\
1 \times (1 \times 1) \arrow[d, "\alpha_{A,0,B}^{T\text{-}\mathbf{Res}} \times (\rho_A^{T\text{-}\mathbf{Res}} \times \mathrm{id}_B^{T\text{-}\mathbf{Res}})"'] & & Z_{A,A} \times Z_{0+B,B} \arrow[dd, "\Box_{A,0+B,A,B}^{T\text{-}\mathbf{Res}}"] \\
Z_{A+(0+B),(A+0)+B} \times (Z_{A+0,A} \times Z_{B,B}) \arrow[d, "Z_{A+(0+B),(A+0)+B} \times \Box_{A+0,B,A,B}^{T\text{-}\mathbf{Res}}"'] & & \\
Z_{A+(0+B),(A+0)+B} \times Z_{(A+0)+B,A+B} \arrow[rr, "\circ^{T\text{-}\mathbf{Res}}"'] & & Z_{A+(0+B),A+B}
\end{tikzcd}$$

Once again, this diagram commutes because the two composites are (parallel) coalgebra morphisms from the coalgebra

$$\mathsf{arr}\big[\,A+(0+B) \xrightarrow{A+\lambda_+} A+B\,\big] \overset{(*)}{=} \mathsf{arr}\big[\,A+(0+B) \xrightarrow{\alpha_+} (A+0)+B \xrightarrow{\rho_+ + B} A+B\,\big]$$
$$: \quad 1 \longrightarrow F_{A+(0+B),A+B}(1)$$

to the final $\zeta_{A+(0+B),A+B}$. The equality $(*)$ is due to the same coherence condition as (26) for the monoidal category $(\mathbf{Set}, +, 0)$; the other details can be easily filled in. This concludes the proof. $\square$

7.5. *Trace structure for resumptions*

A trace/feedback operator $\mathsf{Tr}^{T\text{-}\mathbf{Res}}$ for resumptions is induced by the 'outer' operator Tr for components; this is in exactly the same way as we derived, for example, the tensor $\Box^{T\text{-}\mathbf{Res}}$ from the outer $\Box$. Nevertheless, we will spell out how it is done explicitly.

For arbitrary sets $A, B, C \in T$-**Res**, the *trace* operator

$$\mathsf{Tr}_{A,B,C}^{T\text{-}\mathbf{Res}} \;:\; Z_{A+C,B+C} \longrightarrow Z_{A,B},$$

that is,

$$\mathrm{Hom}_{T\text{-}\mathbf{Res}}(A+C, B+C) \longrightarrow \mathrm{Hom}_{T\text{-}\mathbf{Res}}(A, B),$$

is introduced by the following coinduction diagram:

$$\begin{tikzcd}
F_{A,B}(Z_{A+C,B+C}) \arrow[r, dashed] & F_{A,B}(Z_{A,B}) \\
Z_{A+C,B+C} \arrow[u, "\mathsf{Tr}(\zeta_{A+C,B+C})"] \arrow[r, dashed, "\mathsf{Tr}_{A,B,C}^{T\text{-}\mathbf{Res}}"'] & Z_{A,B} \arrow[u, "\mathrm{final}"', "\zeta_{A,B}"]
\end{tikzcd}$$

Here the operator Tr on the left, acting on $\zeta_{A+C,B+C}$, is the trace operator for components from Definition 6.1.

It is again straightforward to prove that $\mathsf{Tr}^{T\text{-}\mathbf{Res}}$ satisfies the trace axioms: each axiom is essentially reduced to the corresponding axiom on Tr for components. We will prove the post-composition/tightening axiom (21) as an example. This amounts to showing that the following diagram is commutative:

$$\begin{tikzcd}
Z_{A+C,B+C} \times Z_{B,D} \arrow[r, "\mathsf{Tr}^{T\text{-}\mathbf{Res}}_{A,B,C} \times Z_{B,D}"] \arrow[d, "\cong"'] & Z_{A,B} \times Z_{B,D} \arrow[r, "\circ^{T\text{-}\mathbf{Res}}_{A,B,D}"] & Z_{A,D} \\
Z_{A+C,B+C} \times (Z_{B,D} \times 1) \arrow[d, "Z_{A+C,B+C} \times (Z_{B,D} \times \mathrm{id}^{T\text{-}\mathbf{Res}}_{C})"'] & & \\
Z_{A+C,B+C} \times (Z_{B,D} \times Z_{C,C}) \arrow[d, "Z_{A+C,B+C} \times \Box^{T\text{-}\mathbf{Res}}_{B,C,D,C}"'] & & \\
Z_{A+C,B+C} \times Z_{B+C,D+C} \arrow[rr, "\circ^{T\text{-}\mathbf{Res}}_{A+C,B+C,D+C}"'] & & Z_{A+C,D+C} \arrow[uuu, "\mathsf{Tr}^{T\text{-}\mathbf{Res}}_{A,D,C}"']
\end{tikzcd} \tag{27}$$

The composite on the top row is a coalgebra morphism

$$\begin{aligned}
& \mathsf{Tr}(\zeta_{A+C,B+C}) \ggg \zeta_{B,D} \\
\xrightarrow{\mathsf{Tr}^{T\text{-}\mathbf{Res}}_{A,B,C} \times Z_{B,D}} \; & \zeta_{A,B} \ggg \zeta_{B,D} \\
\xrightarrow{\circ^{T\text{-}\mathbf{Res}}_{A,B,D}} \; & \zeta_{A,D}\,;
\end{aligned}$$

where the first arrow is a coalgebra morphism from the definition of $\mathsf{Tr}^{T\text{-}\mathbf{Res}}$ and the functoriality of $\ggg$ (Lemma 4.2 (4)) and the second follows from the definition of $\circ^{T\text{-}\mathbf{Res}}$.

The other composite in (27) is a morphism between the same coalgebras:

$$\begin{aligned}
& \mathsf{Tr}(\zeta_{A+C,B+C}) \ggg \zeta_{B,D} \\
\xrightarrow[\cong]{(*)} \; & \mathsf{Tr}\big[\,\zeta_{A+C,B+C} \ggg \big(\zeta_{B,D} \,\Box\, \mathsf{arr}(\mathrm{id}_C)\big)\,\big] \\
\xrightarrow{Z_{A+C,B+C} \times (Z_{B,D} \times \mathrm{id}^{T\text{-}\mathbf{Res}}_{C})} \; & \mathsf{Tr}\big[\,\zeta_{A+C,B+C} \ggg (\zeta_{B,D} \,\Box\, \zeta_{C,C})\,\big] \\
\xrightarrow{Z_{A+C,B+C} \times \Box^{T\text{-}\mathbf{Res}}_{B,C,D,C}} \; & \mathsf{Tr}(\zeta_{A+C,B+C} \ggg \zeta_{B+C,D+C}) \\
\xrightarrow{\circ^{T\text{-}\mathbf{Res}}_{A+C,B+C,D+C}} \; & \mathsf{Tr}(\zeta_{A+C,D+C}) \\
\xrightarrow{\mathsf{Tr}^{T\text{-}\mathbf{Res}}_{A,D,C}} \; & \zeta_{A,D}.
\end{aligned}$$

Here the isomorphism $(*)$ is due to the post-composition naturality for Tr – see Section 6.3. The last morphism is the definition of $\mathsf{Tr}^{T\text{-}\mathbf{Res}}$. The other arrows are also component morphisms; here the functoriality of Tr (Lemma 6.2) is crucial. Recall that the functor

$$\mathsf{Tr} : \mathbf{Comp}(T, A + C, B + C) \to \mathbf{Comp}(T, A, B)$$

acts on arrows as the identity. We conclude, by the finality of $\zeta_{A,D}$, that diagram (27) commutes.

The other axioms can be verified in the same manner to complete the proof of Theorem 7.4.

8. Concluding remarks

This paper is part of an ongoing line of research into the mathematical (coalgebraic) foundations of components as basic building blocks in computing (see Szyperski (1998) for a wider perspective). Obviously, connections to existing component languages like Reo (Arbab 2004; Baier *et al.* 2006) need to be explored. There are also several directions for further, more mathematically oriented, research on coalgebraic components. We will briefly mention two such avenues, involving duality and dynamic logic.

For specific monads, such as powerset on the category of sets or the identity on the category of Hilbert spaces, the associated Kleisli category carries a dagger operator † that commutes with the tensor and biproduct. By means of this dagger, we can define a duality operator

$$\begin{array}{ccc} \mathbf{Comp}(T, A, B)^{\mathrm{op}} & \xrightarrow{\quad(-)^{\star}\quad} & \mathbf{Comp}(T, B, A) \\ \left(X \otimes A \xrightarrow{c} X \otimes B\right) & \longmapsto & \left(X \otimes B \xrightarrow{c^{\dagger}} X \otimes A\right) \end{array}$$

which satisfies, for instance, $(c \parallel d)^{\star} = (c^{\star} \parallel d^{\star})$. Such a duality introduces a form of reversible computation that may be useful in capturing aspects of quantum computing coalgebraically – see also Abramsky (2009).

Another interesting topic for further research is how to extend modal logic for coalgebras to a dynamic logic (see, for example, Goldblatt (1992)) for coalgebraic components. In such a logic, we expect basic compositionality properties (see also Klin (2009)), so that, for instance, $\Box_{c \ggg d}, \Box_{c \Box d}$, and so on, can be expressed in terms of $\Box_c$ and $\Box_d$.

Acknowledgements

We would like to thank Masahito Hasegawa, Chris Heunen and Ana Sokolova for fruitful discussions, Micah Blake McCurdy for his inspiring work and Shin-ya Katsumata for pointing out the relevance of Micah's work. We would also like to thank two anonymous referees for their useful comments.

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